$\boxed{\log} := \log_2$

H①

$\log 2^x = x$

$\log 1 = 0$ $\qquad 2^0 = 1$

$\log 2 = 1$ $\qquad 2^1 = 2$

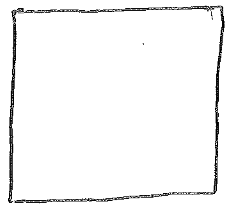
$\log 4 = 2$ $\qquad 2^2 = 4$

$\log 8 = 3$ $\qquad 2^3 = 8$

$\log \frac{1}{2} = -1$ $\qquad 2^{-1} = \frac{1}{2}$

$\log \frac{1}{4} = -2$ $\qquad 2^{-2} = \frac{1}{4}$

$\log \frac{1}{8} = -3$ $\qquad 2^{-3} = \frac{1}{8}$

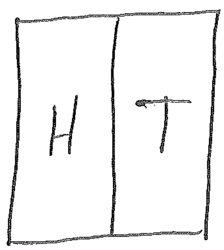


1 unit area

subsets called **events**

---

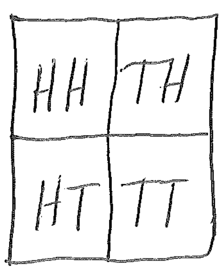
two events: "H" & "T"



models one coin flip

---

four events: "HH", "HT", "TH" "TT"



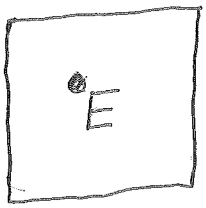
models 2 coin flips

---

partition : null intersections

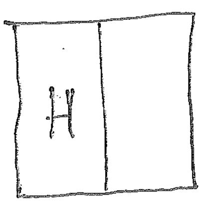
& overall union

$\boxed{\text{info}}$ in an event $:= -\log(\text{its area})$



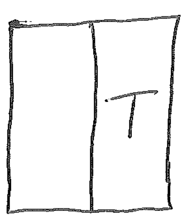
models "the sun rose in the east"

info $= -\log 1 = -0 = 0$



models "heads"

info $= -\log \frac{1}{2} = -(-1) = 1$



models "tails"

info $= -\log \frac{1}{2} = -(-1) = 1$

__expected__ info : $(50\%)\cdot 1 + (50\%)\cdot 1 = (100\%)\cdot 1 = 1$

$\boxed{\text{uncertainty}} := \text{info}$

$50\% = \frac{50}{100} = \frac{1}{2}$

$100\% = \frac{100}{100} = 1$

"sun rose in east"   uncertainty $= 0$

$\boxed{\text{entropy}}$ of a partition := expected uncertainty (info)

HH	TH
HT	TT

$-\log \frac{1}{4} = -(-2) = 2$

$$\text{entropy} = (25\%)\cdot 2 + (25\%)\cdot 2 + (25\%)\cdot 2 + (25\%)\cdot 2$$
$$= (100\%)\cdot 2 = 2$$

<table>
<tr><td>A</td><td rowspan="2">C</td></tr>
<tr><td>B</td></tr>
</table>

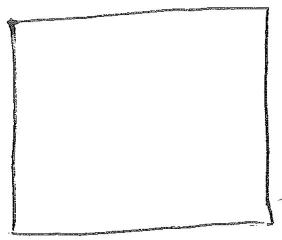
uncertainty in $A = -\log \frac{1}{4} = 2$

uncertainty in $B = -\log \frac{1}{4} = 2$

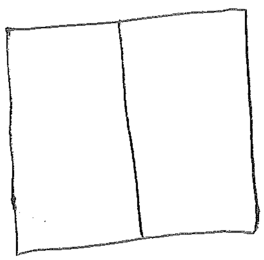
uncertainty in $C = -\log \frac{1}{2} = \underline{1}$

$$\text{entropy} = (25\%)\cdot 2 + (25\%)\cdot 2 + (50\%)\cdot \underline{1}$$
$$= (50\%)\cdot 2 + (50\%)\cdot \underline{1} = 1.5$$

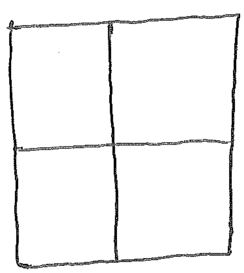
H0



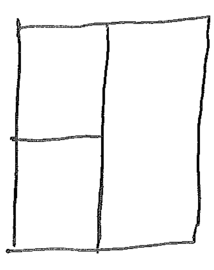
entropy = 0 (1 set)



entropy = 1 (2 sets)

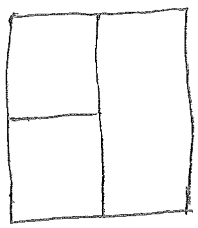


entropy = 2 (4 sets)

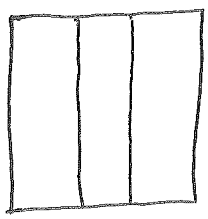


entropy = 1.5 (3 sets)

Q: Using 3 sets, how can we maximize $\underbrace{\text{entropy}}_{\text{expected information}}$?



entropy $= 1.5$ (more?)



$$\text{entropy} = \frac{1}{3}\cdot\left(-\log\frac{1}{3}\right)$$
$$+ \frac{1}{3}\cdot\left(-\log\frac{1}{3}\right)$$
$$+ \frac{1}{3}\cdot\left(-\log\frac{1}{3}\right)$$
$$= -\log\frac{1}{3} \doteq 1.585 \quad \text{(more?)}$$

Def Let n be a positive integer

Let $p_1, \dots, p_n \geq 0$. Assume $p_1 + \cdots + p_n = 1$

Then $\boxed{H(p_1, \dots, p_n)} := p_1 \cdot (-\log p_1)$
$$+ p_2 \cdot (-\log p_2)$$
$$+ \cdots$$
$$+ p_n \cdot (-\log p_n)$$

H7

Focus on $n=2$ (2 set partitions)

Let $p, q \geq 0$ Assume $p+q=1$

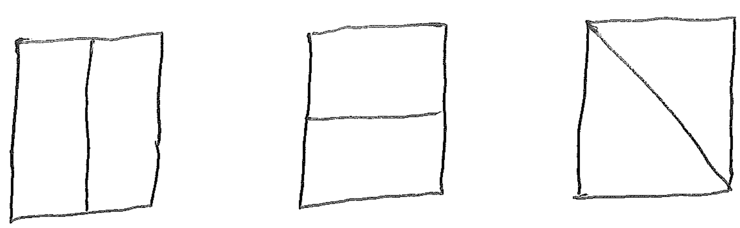
$$H(p,q) = -p\cdot(\log p) - q\cdot(\log q)$$

Constrained optimization problem:

Max $-p\cdot(\log p) - q\cdot(\log q)$

subject to $p, q \geq 0$ and $p+q=1$

Solution: $p=q=\frac{1}{2}$



(All are partitions with two sets that maximize entropy. Many more.)

Focus on $n=3$ (3 set partitions)

Let $p,q,r \geq 0$. Assume $p+q+r=1$

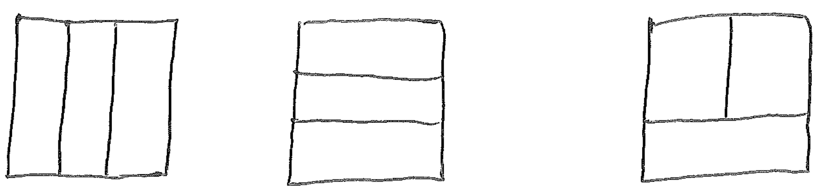
$$H(p,q,r) = -p\cdot(\log p) - q\cdot(\log q) - r\cdot(\log r)$$

Constrained optimization problem:

$$\text{Max } -p\cdot(\log p) - q\cdot(\log q) - r\cdot(\log r)$$

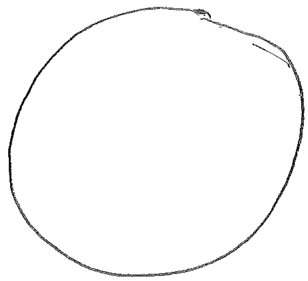
subject to $p,q,r > 0$ and $p+q+r=1$

Solution: $p=q=r=\frac{1}{3}$



(All are partitions with three sets that maximize entropy.

Many more.)



$\leftarrow 10^{23}$ (atoms) particles of hydrogen
(≈ 1 gallon)

Model of kinetic energy distribution:

Quantized energy: 0, 1, 3 —

Total energy: 10^{23}

Random interactions

Each interaction transfer
1 unit of energy from loser to winner
Sometimes 0

Start with: Fair distribution

meaning each particle has 1 unit of energy.

Boltzmann's Qs Asymptotics?

$$\lim_{n \to \infty} \left(\Pr[\text{fair distribution}] \text{ after } n \text{ interactions} \right)$$

$(\Pr[\text{fair distribution}]$ after 0 interactions$) = 100\%$

$(\Pr[\text{fair distribution}]$ after 1 interaction$) = 0\%$

$(\Pr[\text{fair distribution}]$ after 2 interactions$) > 0$

3

4

5

⋮

limit?

Change to:

100 islanders

100 coconuts

Random interactions

Each interaction transfers
1 coconut from loser to winner
Sometimes 0

Start with: Fair distribution
meaning each islander has 1 coconut

Q: $\lim_{n\to\infty}$ (Pr[fair distribution] after n interactions)

Change to: 3 islanders
3 coconuts

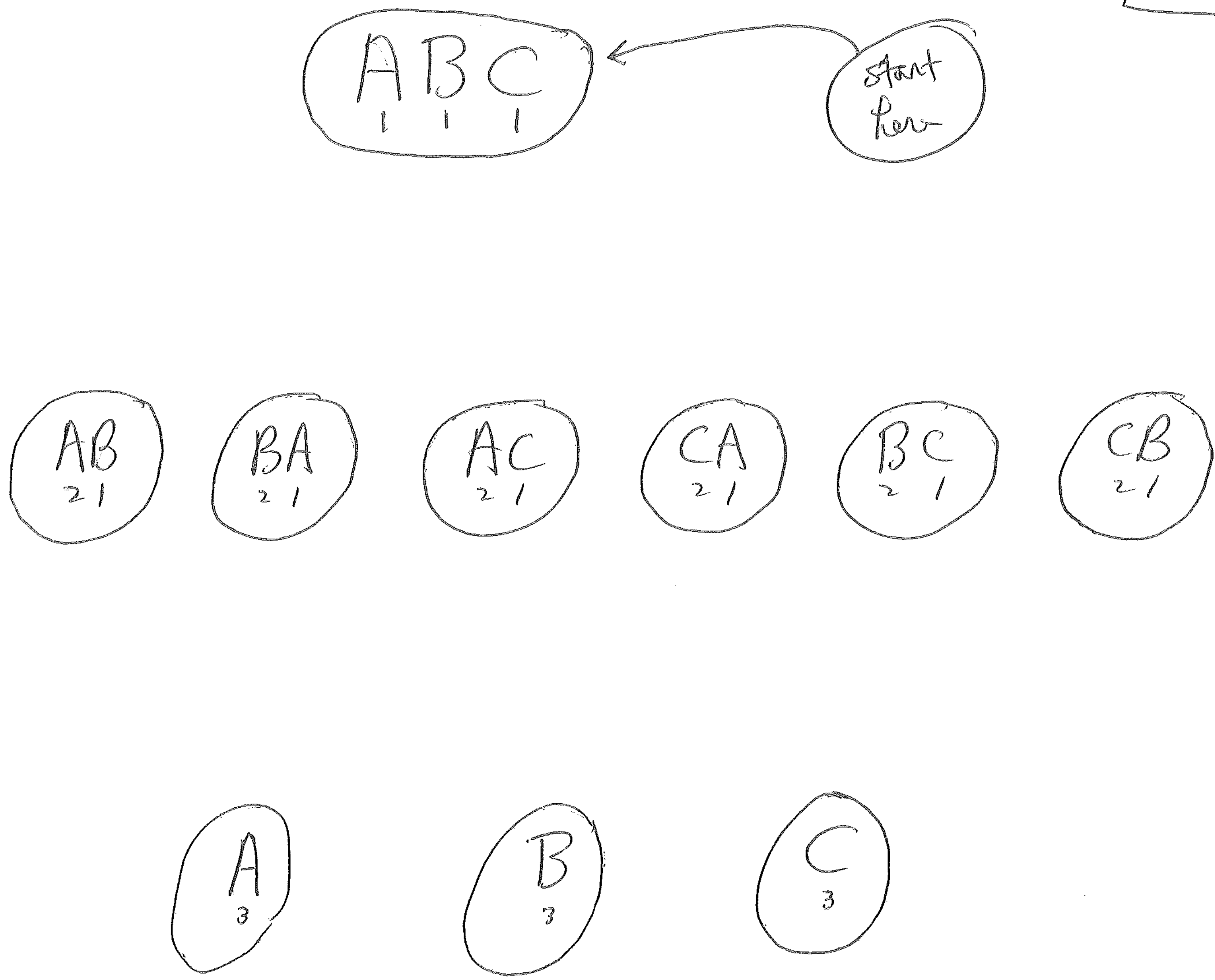
H 12

Islanders: A, B, C

$A_1B_1C_1$:	A1	B1	C1	(Fair)
A_2B_1:	A2	B1	C0	
B_2A_1:	A1	B2	C0	
A_2C_1:	A2	B0	C1	
C_2A_1:	A1	B0	C2	
B_2C_1:	A0	B2	C1	
C_2B_1:	A0	B1	C2	
A_3:	A3	B0	C0	
B_3:	A0	B3	C0	
C_3:	A0	B0	C3	

10 possible distributions

(In <u>all but one</u>, someone has <u>NO</u> coconuts)



Start: ~~[illegible]~~ ABC 1 1 1

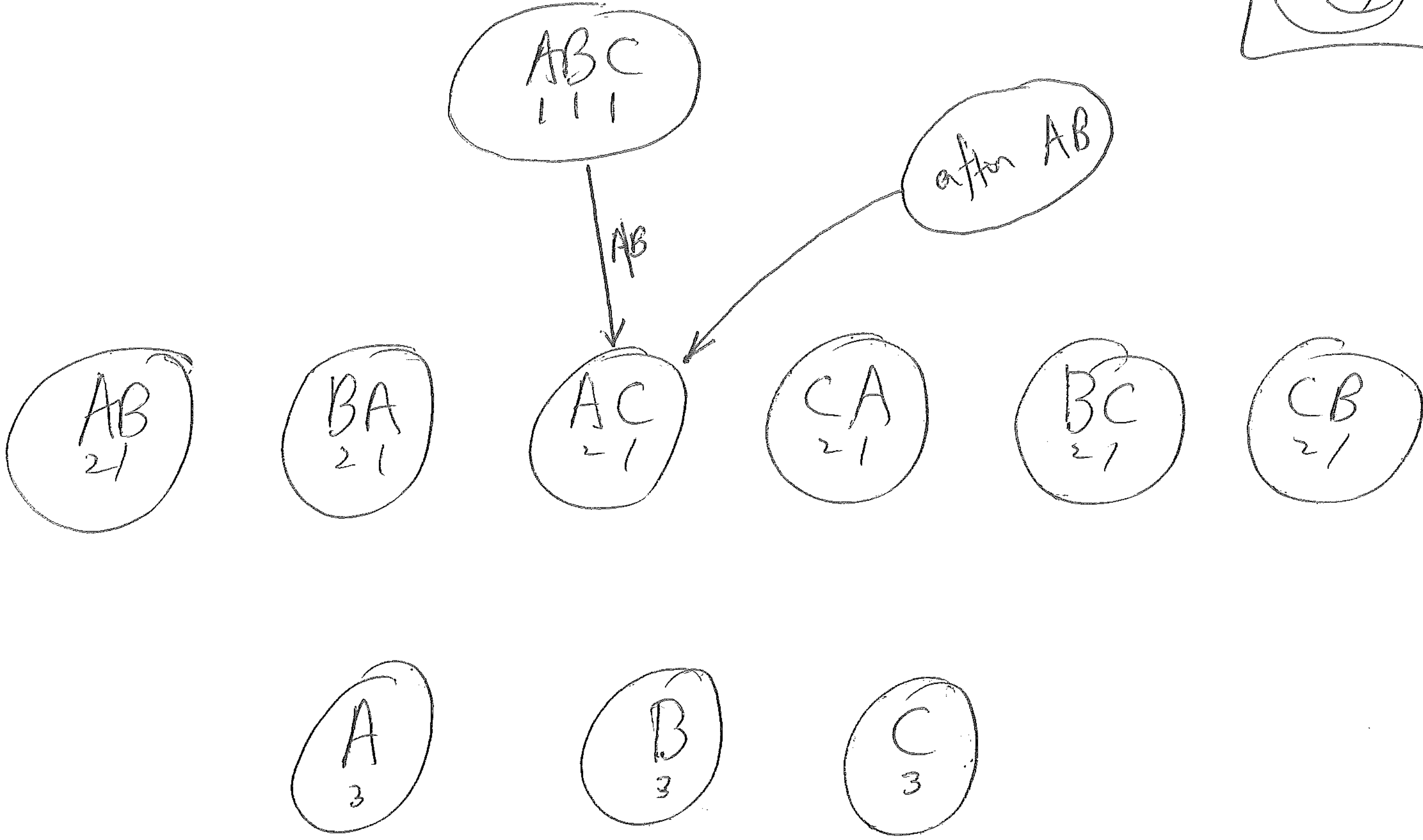
Interactions: A|B, B|A, A|C, C|A, B|C, C|B

Each interaction has probability $1/6$

Suppose A|B occurs.

Go to AC 2 1

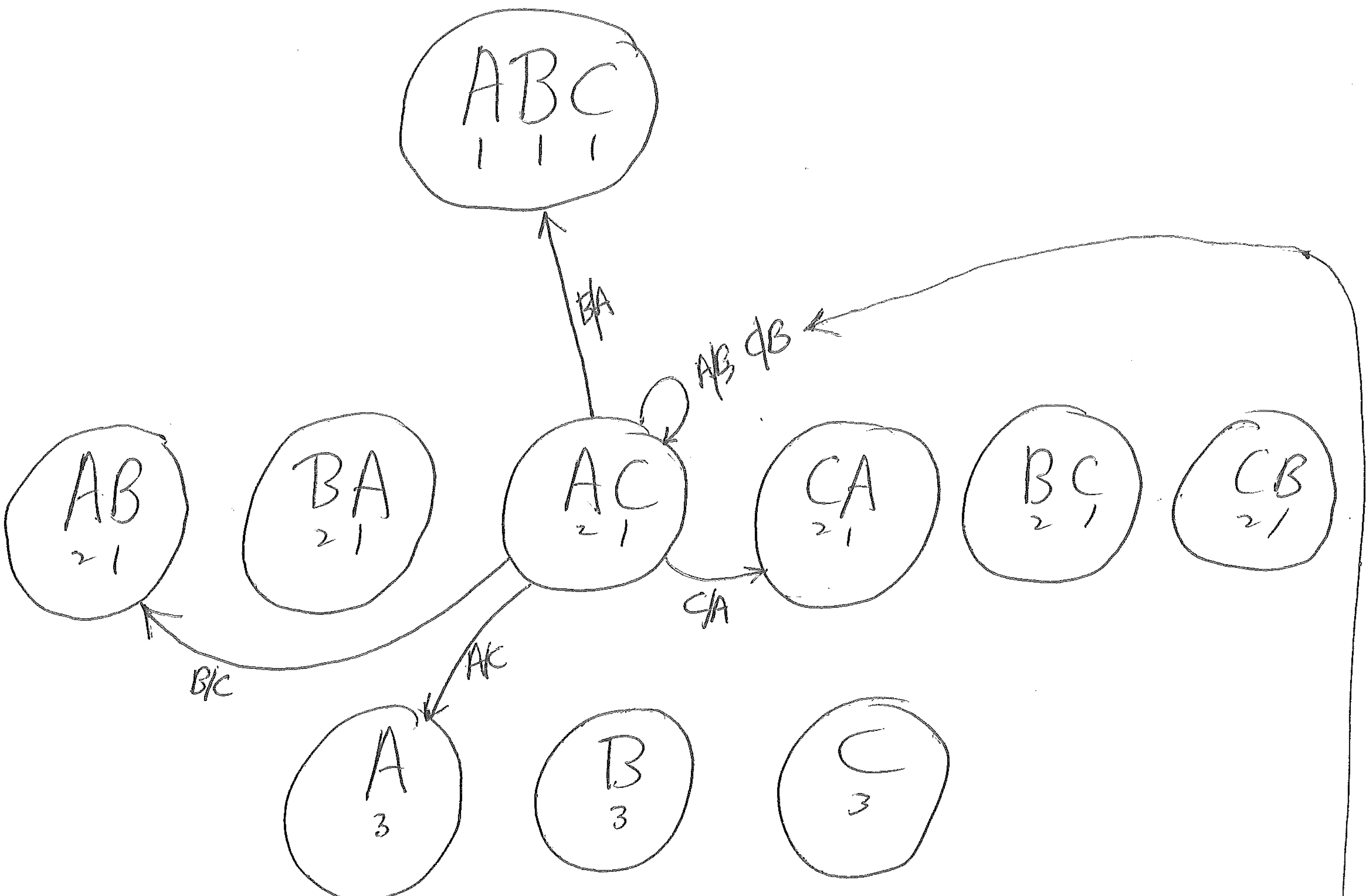
Let's show that one transition (interaction)



Interactions: A|B, B|A, A|C, C|A, B|C, C|B

Each interaction has probability $1/6$

Let's show them all



HW: for each state (=distribution)

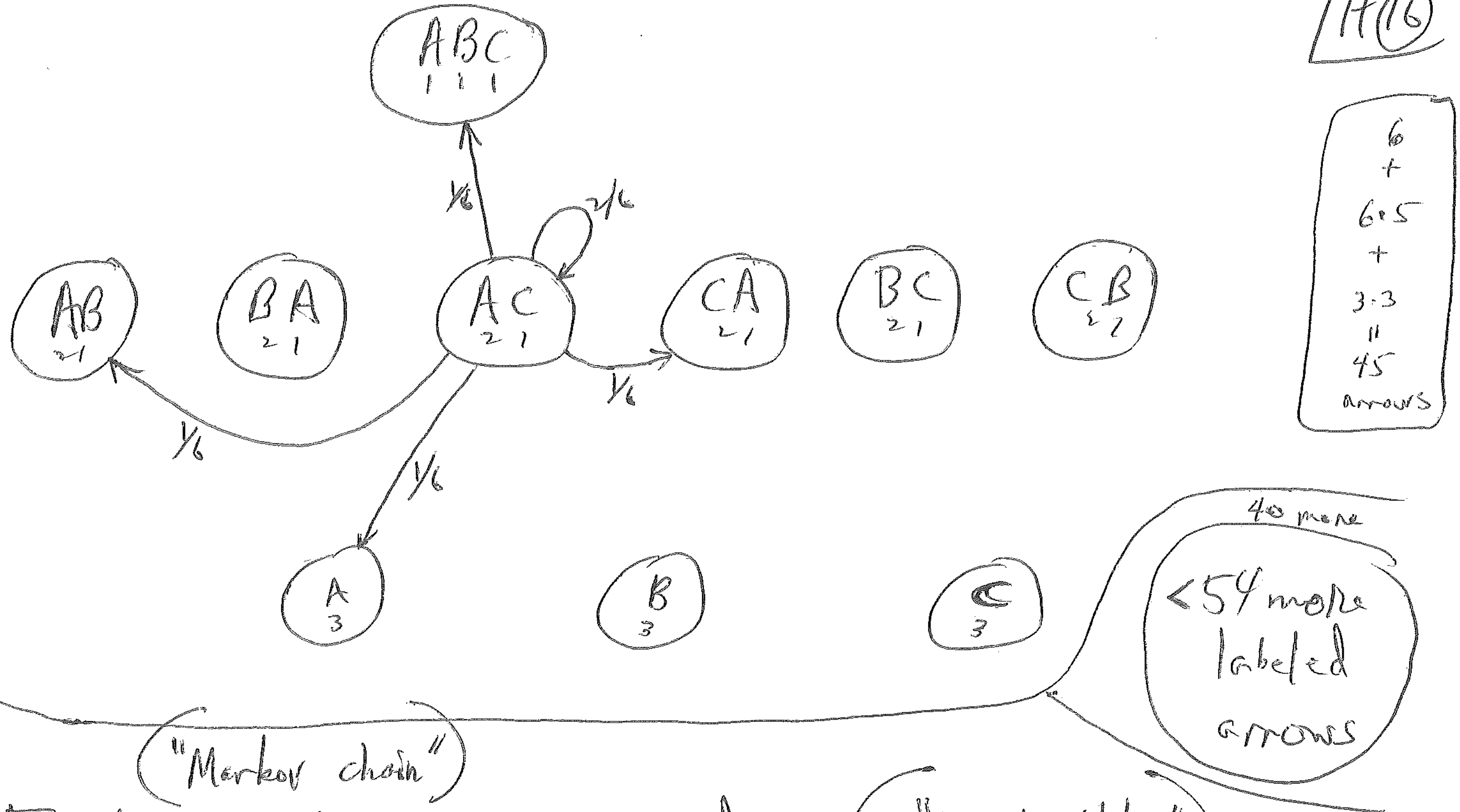
show all 6 transitions (=interactions)

10 states ∴ 60 ~~arrows~~ interactions

$\boxed{\text{45 arrows}}$

Label each arrow with its probability

NOTE: AB CB ⇒ 2/6



Final graph is connected ("irreducible")

There are self-arrows

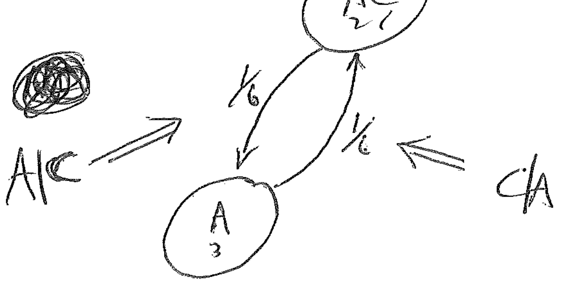
All the other arrows are "paired"

meaning:

For each arrow from one state to another there is an opposite arrow with same probability (label)

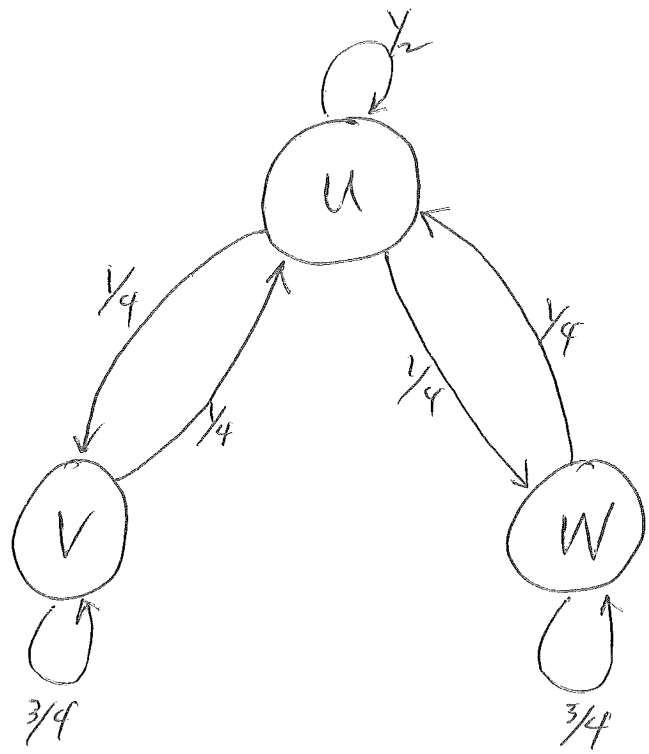
---

e.g.



The final Markov chain is "symmetric"

Simpler MC



This MC is irreducible (connected)

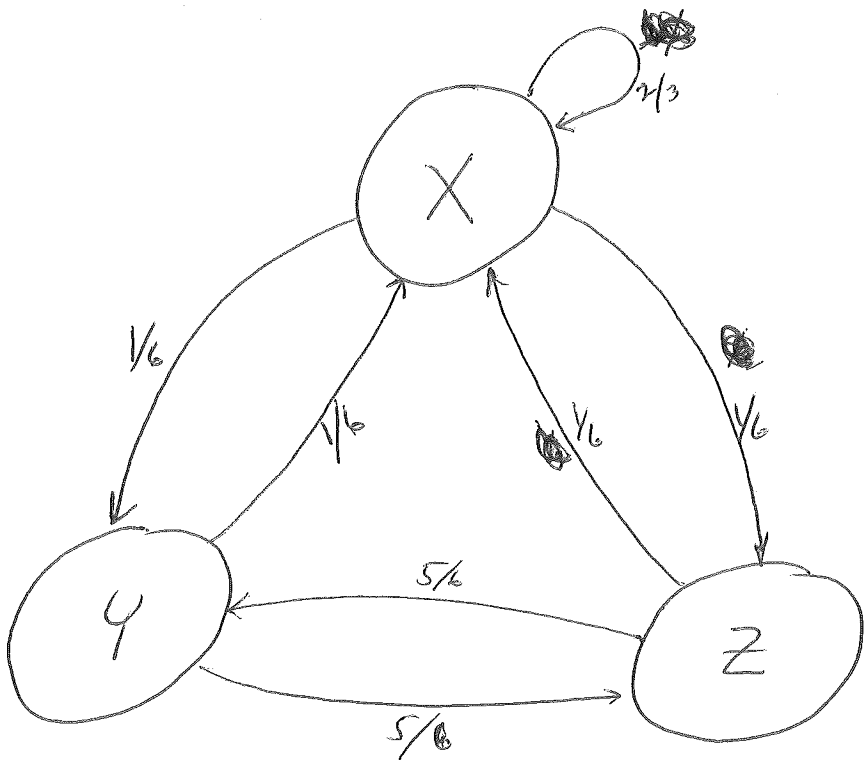
& has self-arrows (at least one)

& is symmetric

Start at U

Q: $\lim_{n \to \infty} \left(\Pr[U] \text{ after } n \text{ transitions} \right)$

Another MC



This MC is irreducible (connected)
& has self-arrows (at least one)
& is symmetric

Start at Z

Q: $\lim_{n \to \infty}$ (Pr [Y] after n transitions)

Perron-Frobenius

Let G be a MC with k states

Assume G is irreducible (connected)

& has self-arrows (at least one)

& is symmetric.

Let S, T be states

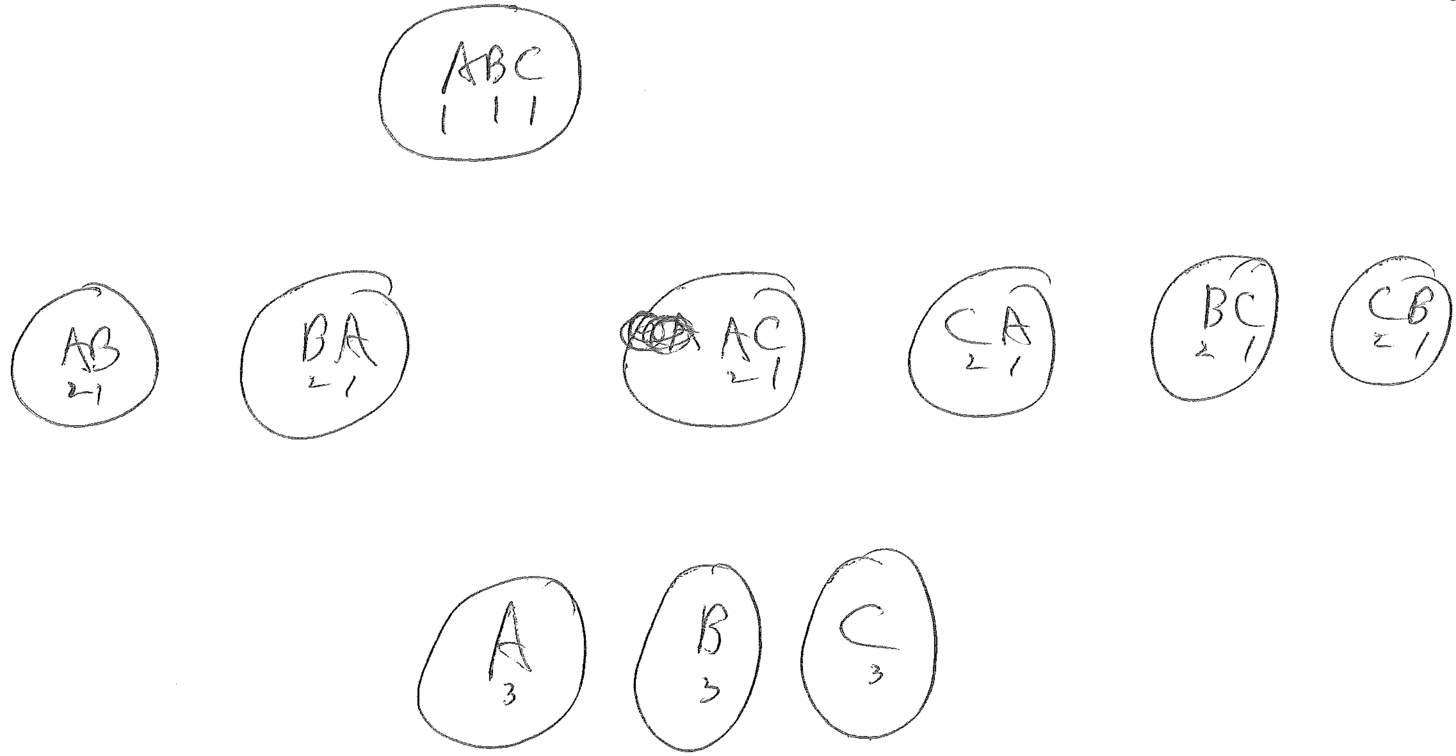
Start at S

Then $\lim_{n \to \infty} \left(\Pr[T] \text{ after } n \text{ transitions} \right) = \frac{1}{k}$

$$\lim_{n \to \infty} \left(\Pr[Y] \text{ after } n \text{ transitions} \right) = \frac{1}{3}$$

$$\lim_{n \to \infty} \left(\Pr[U] \text{ after } n \text{ transitions} \right) = \frac{1}{3}$$

$$\lim_{n \to \infty} \left(\Pr[ABC] \text{ after } n \text{ transitions} \right) = \frac{1}{10}$$



Fill in all 60 labeled arrows

Start at ABC (1 1 1) "fair distribution"

limiting distribution has $\frac{1}{10}$ at every state

limiting distribution has <u>maximal entropy</u>

$$H(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) = -p_1(\log p_1) - \cdots - p_{10}(\log p_{10})$$

max at $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = p_9 = p_{10} = \frac{1}{10}$