Discussion of Chapter 8

1 Introduction

We will discuss some of the ideas in Chapter 8 of our text [4].

We will work to some extent in reverse order, starting with a complicated financial model and then simplifying it to get a Markov chain, which we then analyze further.

2 Building on the Heath-Jarrow-Morton Model

Here we study material from Section 8.7 of our text. One of our goals is to derive the condition on page 253, which guarantees that a discounted defaultable bond price process is a martingale with respect to the given probability measure for the model. We also study this particular model to give a specific context for the ideas. It is only one of the possible settings which are considered in the text.

We consider a time interval $[0,T]$, for some fixed $T > 0$.

2.1 The Default-Free Case

We recall some of the standard setting used earlier in the text.

Following [1], in the *default-free* case the text considers *forward rates* $f(t,s)$, for $0 \le t \le s \le T$. According to [1], $f(t,T)$ "corresponds to the rate one can contract for at time t, on a riskless loan that begins at date T and is returned an instant later". The quantities $f(t,s)$ are random variables, in general.

A default-free bond which matures at time s and pays 1 at time s has (or perhaps one should say is *assumed* to have) a price given by

$$B(t,s) \equiv e^{-\int_t^s f(t,u)\,du} \tag{1}$$

at time t. We will need to take an expectation of this price to compare it with experimental data.

If we choose to think of $B(t,s)$ as the primary concept, we can define $f(t,s)$ by

$$f(t,s) = -\frac{\partial}{\partial s} \log B(t,s) \tag{2}$$

for all $s \in [0,T]$ and all $t \in [0,s]$.

Remark Differentiating $B(t,s)$ with respect to t is much harder than differentiating with respect to s, because t appears inside the integral, and $f(t,s)$ is not differentiable with respect to t in the calculus sense. This is one of the problems which is solved in [1], as we will see below. They are able to show that following the same pattern of differentiation under integral sign used in calculus gives the correct formula.

Given a family $f(t,s)$, we let

$$r(t) = f(t,t). \tag{3}$$

This is referred to as the *spot rate*, and it appears that it can actually be observed.

It seems that the main information we use from the default-free case is the spot rate $r(t)$.

Now we consider the defaultable bond case.

2.2 Behavior in between rating changes

We suppose there are $K-1$ ratings categories, which we label by $k = 1, \ldots, K-1$. The first ingredient in the overall model is the time behavior of a typical bond in category k during periods when no drastic changes take place in the state of the issuing company, i.e. during periods when there is *no change in rating*. For each k we consider a (fictitious) bond, called the *benchmark bond*, which never changes its rating, so that its behavior over the whole time interval is of the non-drastic sort.

Thus we assume that for each $k \in \{1, \ldots, K-1\}$, there is a forward rate $\overline{f}_k(t,s)$ (with a corresponding spot rate $\overline{r}_k(t)$). This is the benchmark rate for rating category k.

A benchmark bond in category k which pays 1 at time s is considered to be worth

$$\overline{B}_k(t,s) \equiv e^{-\int_t^s \overline{f}_k(t,u)\,du} \tag{4}$$

We might recall the comment made by Schönbucher on page 51: "Unfortunately, we cannot observe the prices of such a set of zero-recovery defaultable zero-coupon bonds in real markets…". This appears to apply to both $B(t,T)$ and $\overline{B}_k(t,T)$.

Up to this point we have not specified the statistical behavior of the random forward rates. We assume now that for each fixed $s \in [0,T]$, for $t \in [0,s]$ the quantity $\overline{f}_k(t,s)$ has the following stochastic differential respect to t:

$$d\,\overline{f}_k(t,s) = \overline{\alpha}_k(t,s)\,dt + \sum_{i=1}^{n} \overline{\sigma}_{i,k}(t,s) \cdot dW^i(t), \tag{5}$$

where $W^i(t)$, $i = 1, \ldots, n$ are *independent* Wiener processes. For technical reasons we assume that the $\overline{\sigma}_{i,k}(t,s)$ are predictable stochastic processes.

The text describes the benchmark bonds as defaultable, but notes that "they never change their rating, nor are they affected by defaults". This suggests that if they were real bonds they would be priced at the default-free price, so we should think of them as only an *ingredient* in a real bond.

Remark on nonnegative forward rates From the context, it is required (page 53 of the text) that the functions $\overline{f}_k(t,T)$ should be nonnegative, i.e. that we must have nonnegative solutions to (5) with $s = T$. Apparently there is no natural general condition to guarantee this. The initial value $\overline{f}_k(0,T)$, the sign of $\overline{\alpha}_k(t,s)$ and the size of $\overline{\sigma}_{i,k}(t,s)$ are all relevant.

2.3 The actual bond

We now consider the model for an actual defaultable bond, as opposed to the benchmark bonds just described.

Let $\overline{B}(t,T)$ be the price at time t of a defaultable bond which pays 1 at time T. This is the quantity we wish to study.

Let $R(t)$ denote the rating of the bond at time t.

We agree on a constant $q > 0$, which is the fraction of value lost by a bond during a default. If $q = 1$ then the value of the bond is zero after a default. This is the *zero recovery* case.

2.3.1 Rating changes and defaults

The values of $R(t)$ lie in the set $\{1, \ldots, K-1\}$.

To have a very general setting, we can say that there are markers (k,ℓ), $k,\ell \in \{1,\ldots,K-1\}$, and $k \in \{1,\ldots,K-1\}$. The marker (k,ℓ) symbolizes a *ratings transition* from k to ℓ and the marker k symbolizes a default of from rating state k.

That is, we assume there is a jump measure μ such that if $\mu((k,\ell),\{t\}) = 1$, and if $R(t-) = k$, then $R(t) = \ell$, and such that if $\mu(k,\{t\}) = 1$, and if $R(t-) = k$, then a default occurs at time t.

We assume the jump measure μ has a compensator ν such that

$$\nu((k,\ell), dt) = \lambda_{k,\ell}(t)\, dt, \tag{6}$$

and

$$\nu(k, dt) = \psi_k(t)\, dt, \tag{7}$$

where the *intensities* (or *rates*) $\lambda_{k,\ell}(t)$ and $\psi_k(t)$ are predictable random process.

More concretely, if $\lambda_{k,\ell}(t)$ is a constant which does not depend on omega then $R(t)$ is a time-inhomogeneous Markov chain. If in addition $\lambda_{k,\ell}(t)$ does not depend on t then $R(t)$ is a time-homogeneous Markov chain. The Markov chain is discussed further below.

We should probably think of the Markov chain case to provide intuition and regard the general case as an extension in which we allow more complex rates, which could, for example, depend at time t on the whole history of all the bond prices up to time t.

2.3.2 Remark: No state to represent default

In some other models considered in Section 8 of our text, there are K states, and state K represents the default state. We don't have such a state in our representation, because we want to allow for some value to remain after default. As Schönbucher puts it (on page 252), after a default "the face value of the defaulted bond is reduced (multiplied by $1 - q < 1$), and the obligor continues his life in the pre-default rating class".

If we are in the zero recovery case, we can add a state K to represent a company in default. In Markov chain terms this state is an *absorbing* state, meaning that once it is entered, the process never leaves this state. The average rate per unit time at which the process moves from a state $k \in \{1, \ldots, K-1\}$ to state K is just the average rate per unit time at which a default occurs for a company in state k. In terms of usefulness, having an extra state to represent default does not increase the applicability of our discussion, but it permits more familiar terminology in the zero recovery case.

2.3.3 The price of the defaultable bond

Let $N(t)$ denote the number of defaults the company which issues the bond has suffered by time t. Define the surviving fraction of value at time t to be

$$Q(t) \equiv (1-q)^{N(t)}. \tag{8}$$

By definition, the price of our defaultable bond at time t is

$$\overline{B}(t,T) \equiv Q(t)\,\overline{B}_{R(t)}(t,T). \tag{9}$$

This price only takes account of possible future defaults through the forward rates $\overline{f}_k(t,T)$.

3 The differential of a benchmark bond price

We will eventually want to study the compensator of the process $\overline{B}(t,T)$. The key step for that is to find the stochastic differential $d\overline{B}_k(t,T)$ for each $k \in \{1, \ldots, K-1\}$. For that purpose we use a nice argument from [1]. In that paper it is shown that one can take a stochastic differential inside the integral with respect to a parameter. Thus the stochastic differential turns out to be exactly what one would expect from a purely formal manipulation.

$$
\begin{aligned}
&\int_t^T \overline{f}_k(t,u)\,du - \int_t^T \overline{f}_k(0,u)\,du = \\
&\qquad \int_t^T \left(\int_0^t \overline{\alpha}_k(v,u)\,dv\right)du + \int_t^T \left(\int_0^t \sum_{i=1}^n \overline{\sigma}_{i,k}(v,u)\cdot dW^i(v)\right)du \\
&= \quad \int_0^t \left(\int_t^T \overline{\alpha}_k(v,u)\,du\right)dv + \int_0^t \sum_{i=1}^n \left(\int_t^T \overline{\sigma}_{i,k}(v,u)\,du\right)\cdot dW^i(v).
\end{aligned}
$$

Here we used the fact that we can exchange the order of integration under fairly general conditions, even when one of the integrals is a stochastic integral.

We then have

$$
\begin{aligned}
&\int_t^T \overline{f}_k(t,u)\,du - \int_t^T \overline{f}_k(0,u)\,du = \\
&\qquad \int_0^t \left(\int_v^T \overline{\alpha}_k(v,u)\,du\right)dv + \int_0^t \sum_{i=1}^n \left(\int_v^T \overline{\sigma}_{i,k}(v,u)\,du\right)\cdot dW^i(v) \\
&- \quad \int_0^t \left(\int_v^t \overline{\alpha}_k(v,u)\,du\right)dv - \int_0^t \sum_{i=1}^n \left(\int_v^t \overline{\sigma}_{i,k}(v,u)\,du\right)\cdot dW^i(v).
\end{aligned}
$$

We note that

$$
\begin{aligned}
&\int_0^t \left(\int_v^t \overline{\alpha}_k(v,u)\,du\right)dv + \int_0^t \sum_{i=1}^n \left(\int_v^t \overline{\sigma}_{i,k}(v,u)\,du\right)\cdot dW^i(v) = \\
&\qquad \int_0^t \left(\int_0^u \overline{\alpha}_k(v,u)\,dv\right)du + \int_0^t \left(\int_0^u \sum_{i=1}^n \overline{\sigma}_{i,k}(v,u)\cdot dW^i(v)\right)du \\
&= \quad \int_0^t \left(\overline{f}_k(u,u) - \overline{f}_k(0,u)\right)du.
\end{aligned}
$$

For brevity, let

$$
\overline{\eta}_k(v,T) = \int_v^T \overline{\alpha}_k(v,y)\,dy, \tag{10}
$$

$$
\overline{\xi}_{i,k}(v,T) = \int_v^T \overline{\sigma}_{i,k}(v,y)\,dy. \tag{11}
$$

Clearly these are just the integrated forms of the drift and the volatility.

Combining our facts, we have

$$
\begin{aligned}
&\int_t^T \overline{f}_k(t,u)\,du - \int_0^T \overline{f}_k(0,u)\,du = \\
&\qquad \int_0^t \left(-\overline{r}_k(v) + \overline{\eta}_k(v,T)\right)dv + \int_0^t \sum_{i=1}^n \overline{\xi}_{i,k}(v,T)\cdot dW^i(v).
\end{aligned} \tag{12}
$$

That is,

$$d\left(-\log\overline{B}_k(t,T)\right)=-\overline{r}_k(t)\ dt+\overline{\eta}_k(t,T)\ dt+\sum_{i=1}^{n}\overline{\xi}_{i,k}(t,T)\cdot dW^i(t)\,. \quad (13)$$

For any continuous semimartingale X, an application of Itô's Lemma shows that

$$d\,e^X=e^X\,d\,X+\frac{1}{2}e^X\,d\langle X,X\rangle. \quad (14)$$

Hence we can easily find $d\overline{B}_k(t,T)$ from (13).

$$\begin{aligned}-\frac{d\overline{B}_k(t,T)}{\overline{B}_k(t,T)} &= -\overline{r}_k(t)\ dt+\overline{\eta}_k(t,T)\ dt\\ &- \frac{1}{2}\sum_{i=1}^{n}\overline{\xi}_{i,k}(t,T)^2\ dt+\sum_{i=1}^{n}\overline{\xi}_{i,k}(t,T)\cdot\ dW^i(t)\,. \quad (15)\end{aligned}$$

4 The compensator for $\overline{B}(t,T)$

Let κ denote the compensator for $\overline{B}(t,T)$. We wish to find $d\,\kappa(t)$. This can be obtained using the properties of compensators given in Chapter 4, using the fact that each change in rating occurs at a stopping time, and we can apply the compensators for the $\overline{B}_k(t,T)$ processes in between the stopping times.

In the discussion here we'll only give a heuristic derivation.

We consider $d\,\kappa(t)$ as the sum of the conditional expected values of all the contributions to $d\,\overline{B}(t,T)$.

From (9) we have

$$\begin{aligned}E\left(d\,\overline{B}(t,T)\,|\mathcal{F}_t\right) &= Q(t)\sum_{k=1}^{K-1}\mathbf{1}_{\{R(t)=k\}}E\left(d\,\overline{B}_k(t,T)\,|\mathcal{F}_t\right)\\ &+ Q(t)\sum_{k=1}^{K-1}\sum_{\ell=1}^{K-1}\mathbf{1}_{\{R(t)=k\}}\lambda_{k,\ell}(t)\left(\overline{B}_\ell(t,T)-\overline{B}_k(t,T)\right)\ dt\\ &- Q(t)\sum_{k=1}^{K-1}\mathbf{1}_{\{R(t)=k\}}q\psi_k(t)\,\overline{B}_k(t,T)\ dt.\end{aligned}$$

From (15) we have

$$\begin{aligned}&E\left(d\,\overline{B}_k(t,T)\,|\mathcal{F}_t\right)=\\ &\quad-\overline{B}_k(t,T)\left(-\overline{r}_k(t)\ dt+\overline{\eta}_k(v,t)\ dt-\frac{1}{2}\sum_{i=1}^{n}\overline{\xi}_{i,k}(v,T)^2\ dt\right). \quad (16)\end{aligned}$$

Thus

$$\begin{aligned}
&E\left(d\,\overline{B}\,(t,T)\,|\mathcal{F}_t\right) = \\
&\qquad Q\,(t)\sum_{k=1}^{K-1}\mathbf{1}_{\{R(t)=k\}}\left(-\overline{B}_k\,(t,T)\right)(-\overline{r}_k\,(t)\;dt \\
&+\quad \overline{\eta}_k\,(v,t)\;dt - \frac{1}{2}\sum_{i=1}^{n}\overline{\xi}_{i,k}\,(v,T)^2\;dt\Bigg) \\
&+\quad Q\,(t)\sum_{k=1}^{K-1}\mathbf{1}_{\{R(t)=k\}}\overline{B}_k\,(t,T)\left(\sum_{\ell=1}^{K-1}\lambda_{k,\ell}\,(t)\left(\frac{\overline{B}_\ell\,(t,T)}{\overline{B}_k\,(t,T)}-1\right)\right)\,dt \\
&-\quad Q\,(t)\sum_{k=1}^{K-1}\mathbf{1}_{\{R(t)=k\}}q\psi_k\,(t)\,\overline{B}_k\,(t,T)\;dt.
\end{aligned}\tag{17}$$

4.1 Conditions for a martingale measure

We require that for each $k \in \{1, \ldots, K-1\}$,

$$\psi_k\,(t)\,q = \overline{r}_k\,(t) - r\,(t)\,. \tag{18}$$

This says that the spot interest rate for the defaultable bond should exceed the spot interest rate for the default-free bond by an amount which compensates exactly for the expected loss due to default during a short period of time.

We also require that for each k,

$$\overline{\eta}_k\,(t,T) = \frac{1}{2}\sum_{i=1}^{n}\overline{\xi}_{i,k}\,(t,T)^2 + \sum_{\ell=1}^{K-1}\lambda_{k,\ell}\,(t)\left(\frac{\overline{B}_\ell\,(t,T)}{\overline{B}_k\,(t,T)}-1\right). \tag{19}$$

Substituting into (17) we obtain

$$\begin{aligned}
&E\left(d\,\overline{B}\,(t,T)\,|\mathcal{F}_t\right) = \\
&\qquad Q\,(t)\sum_{k=1}^{K-1}\mathbf{1}_{\{R(t)=k\}}\overline{B}_k\,(t,T)\,(\overline{r}_k\,(t)\;dt - q\psi_k\,(t))\;dt \\
&=\quad \overline{B}\,(t,T)\,r\,(t)\;dt.
\end{aligned}\tag{20}$$

That is,

$$d\overline{B}\,(t,T) = \overline{B}\,(t,T)\,r\,(t)\;dt + d\,M, \tag{21}$$

where M is a *local martingale*.

We will now consider the *default-free discounted value* of $\overline{B}\,(t,T)$. The default-free discount factor $\beta\,(t)$ is defined by

$$\beta\,(t) = e^{-\int_0^t r(u)\,du}. \tag{22}$$

Obviously the process $\beta\,(t)$ is of bounded variation and $d\,\beta\,(t) = -\beta\,(t)\,r\,(t)$.

For any semimartingales X, Y, where at least one of the semimartingales is continuous, we have the simple product rule as a special case of Itô's Formula:

$$d(XY) = X\,dY + Y\,dX + d\langle X, Y\rangle. \tag{23}$$

It follows that if (21) holds we have

$$\begin{aligned} & d\left(\beta(t)\,\overline{B}(t,T)\right) \\ &= \beta(t)\left(\overline{B}(t,T)\,r(t)\;dt + d\,M\right) - \overline{B}(t,T)\,\beta(t)\,r(t)\;dt \\ &= \beta(t)\,d\,M. \end{aligned} \tag{24}$$

Hence the conditions (18) and (19) imply that the discounted price $\beta(t)\,\overline{B}(t,T)$ of the defaultable bound is a local martingale.

This is almost the desired condition. We still have to verify in some way that the process $\beta(t)\,\overline{B}(t,T)$ is uniformly integrable, for example that it is bounded by an integrable random variable. Then we can deduce that $\beta(t)\,\overline{B}(t,T)$ is a martingale from the fact that it is a local martingale.

4.2 Simplifying the drift condition

In [3], it is noted that we can simplify (19) by differentiating with respect to T and using the analog of (2), i.e.

$$\overline{f}_k(t,s) = -\frac{\partial}{\partial s}\log \overline{B}_k(t,s).$$

This gives

$$\begin{aligned} \overline{\alpha}_k(t,T) &= \sum_{i=1}^{n} \overline{\xi}_{i,k}(t,T)\,\overline{\sigma}_{i,k}(t,T) \\ &+ \sum_{\ell=1}^{K-1} \lambda_{k,\ell}(t)\left(\frac{\overline{B}_\ell(t,T)}{\overline{B}_k(t,T)}\right)\left(\overline{f}_k(t,T) - \overline{f}_\ell(t,T)\right). \end{aligned} \tag{25}$$

Equation (25) is the condition (8.59) given on page 253 of the text.

4.3 Non-uniqueness of martingale measures

There seem to be many adjustable quantities in (18) and (19). For that reason we should not expect a unique martingale measure. Schönbucher discusses similar issues on page 246 in a somewhat different context.

5 Finding expected values in the Markov case

In the context of the previous section, we assume that the process $\beta(t)\,\overline{B}(t,T)$ is a *martingale*. We also assume that the model is such that the rates $\lambda_{k,\ell}(t)$ and $\psi_k(t)$ are *deterministic* i.e. do not depend on ω. Then $R(t)$ is a Markov chain which is time-inhomogeneous in general. We want to see how to calculate the expected value of $\beta(t)\overline{B}(t,T)$ in this case.

Suppose that we are interested in a particular state $k_0 \in \{1,\ldots,K-1\}$. Assume that $P(R(0)=k_0)>0$.

By the martingale property,

$$\begin{aligned} E\beta(T)\overline{B}(T,T)\mathbf{1}_{\{R(0)=k_0\}} &= \int \mathbf{1}_{\{R(0)=k_0\}} E\left(\beta(T)\overline{B}(T,T)\,|\mathcal{F}_0\right)\,dP \\ &= \int \mathbf{1}_{\{R(0)=k_0\}} \beta(0)\overline{B}(0,T)\,dP \\ &= E\mathbf{1}_{\{R(0)=k_0\}}\overline{B}(0,T) \end{aligned}$$

Hence dividing by $P(R(0)=k_0)$ we obtain

$$E\left(\beta(T)\overline{B}(T,T)\,|R(0)=k_0\right) = E\left(\overline{B}(0,T)\,|R(0)=k_0\right). \tag{26}$$

Let's also assume that $r(t)$ is deterministic. We have

$$\beta(T)\overline{B}(T,T) = e^{-\int_0^T r(s)\,ds} Q(T).$$

Then

$$E\left(\beta(T)\overline{B}(T,T)\,|R(0)=k_0\right) = e^{-\int_0^T r(s)\,ds} E\left(Q(T)\,|R(0)=k_0\right). \tag{27}$$

The latter quantity, $E(Q(T)\,|R(0)=k_0)$, can be found by a purely Markov chain calculation, which we will briefly sketch.

We'll assume that $r(t)$, $\lambda_{k,\ell}(t)$ and $\psi_k(t)$ are all continuous functions of time t.

Suppose that $R(t)=k$. For a small positive number h, the probability that $R(t+h)=\ell$ is given by $\lambda_{k,\ell}(t)\,h+\mathrm{o}(h)$. The probability that $R()$ makes more than one change during the interval $[t,t+h]$ is also $\mathrm{o}(h)$.

Again suppose that $R(t)=k$. For a small positive number h, the probability of a default during the time interval $[t,t+h]$ is given by $\psi_k(t)\,h+\mathrm{o}(h)$.

Let $N(t,T)$ denote the number of defaults during $[t,T]$. Let

$$g_k(t) = E\left((1-q)^{N(t,T)}\,|R(t)=k\right). \tag{28}$$

We wish to find $g_{k_0}(0)$. Comparing $g_k(t+dt)$ and $g_k(t)$, we find that

$$-g_k'(t) = -\psi_k(t)\,g_k(t) + \sum_{\ell\neq k}\lambda_{k,\ell}(t)\left(g_\ell(t)-g_k(t)\right). \tag{29}$$

We have "initial" conditions $g_k(T)=1,\ k=1,\ldots,K-1$ for this system of linear differential equations, so a solution is easy.

6 Embedding a transition matrix in a Markov semigroup

From now on, to simplify our discussion we consider the zero recovery case, when $q = 1$, and we add a state K to represent a company in default.

We set $\lambda_{k,K}(t) = \psi_k(t)$ for all $k \in \{1, \ldots, K-1\}$ and we necessarily set $\lambda_{K,k}(t) = 0$ for all $k \in \{1, \ldots, K-1\}$. From now on we will assume that $\lambda_{k,\ell}(t) = \lambda_{k,\ell}$, independent of time.

Rather arbitrary choices for $\lambda_{k,\ell}$ may be consistent with the martingale measure condition but are probably not helpful. It seems to be desirable to choose the rates $\lambda_{k,\ell}$ in a way which is consistent with observed data from bond ratings.

Let Λ denote the $K \times K$ matrix $(\lambda_{k,\ell})$, where we make one more definition and set

$$\lambda_{k,k} = -\sum_{k \neq \ell} \lambda_{k,\ell}.$$

This condition guarantees that the rows of Λ sum to 0. Note that $-\lambda_{k,k}$ is the rate at which probability flows from state k into all the other states. There is an associated semigroup of *Markov operators* $Q(t)$, $t \geq 0$, given by

$$Q(t) = e^{\Lambda t}.$$

The semigroup equation $Q(t+s) = Q(t)Q(s)$ obviously holds, and is sometimes referred to as the Chapman-Kolmogorov equation. Differentiating this equation with respect to t and s immediately gives two related differential equations: $Q' = Q\Lambda = \Lambda Q$, called the forward and backward Chapman-Kolmogorov differential equations. The rate matrix Λ is referred to as the *generator* of the semigroup $Q(t)$.

It is standard to show that

$$[Q(t)]_{k\ell} = P\left(R(t) = \ell | R(0) = k\right).$$

For this reason $Q(t)$ is called the Markov transition matrix for the Markov chain $R()$, and $Q(t)$ is the quantity we need to know to carry out our expectation calculations. We also refer to $(Q(t))_{t \geq 0}$ as the Markov semigroup.

However, we will not have all the data we need to find $Q(t)$ empirically. In Section 8.3 the text considers the situation in which we are given enough information to find, say, $Q(1)$, and then must attempt to find $Q(t)$ for all t, or, equivalently, to find the generator Λ. This is the *embedding* problem.

The paper [2] by Israel, Rosenthal and Wei gives a good overview of the mathematical issues involved in the embedding problem. In general, for a proposed $Q(1)$ there may be *no* semigroup $Q(t)$, or there may be *many* solutions!

6.1 A reasonable alternative

The quantities

$$\gamma_k = \sum_{\ell \neq k} \lambda_{k,\ell}$$

and

$$\rho_{k\ell} = \frac{\lambda_{k,\ell}}{\gamma_k}$$

carry exactly the same information as Λ and are at least as intuitive. γ_k is the rate for the exponential waiting time that the system spends in state k before jumping to another state. $(\rho_{k\ell})$ is the "routing matrix" of transition probabilities that specifies how likely it is that the system moves to state ℓ when it leaves state k. We would probably use γ_k and $\rho_{k\ell}$ in carrying out a numerical simulation of the Markov chain.

Assume that we believe that ratings changes actually obey a time-homogeneous Markov chain model. Then, if we are given *time-stamped* data for some time period (i.e. the history of ratings changes and the times at which they occur), it appears to be trivial to find a natural estimate for γ_k and $\rho_{k\ell}$. That is, all the mathematical subtleties in the embedding problem disappear.

Both the text, in Section 8.4, and [2] make comments about this approach, and it appears to be feasible, at least in many cases.

On page 238 the text says that for the time-stamped approach exact times of ratings transitions must be known. Perhaps exactness is not meant literally here. More precisely, if one *truly* believes that a time-homogeneous Markov model is appropriate, then it seems that only the *sequential* record of the transitions, not the exact times, is sufficient. On the other hand, the exact details of the time record might cause a less fanatical statistician to *reject* the time-homogeneous model, so they are relevant in that sense.

References

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[4] Schönbucher, P., *Credit Derivatives Pricing Models*, Wiley, Chichester 2003.