Stochastic calculus for jump processes

Simple predictable processes:

$$\phi_t = \phi_0 1_{t=0} + \sum_{i=0}^{n} \phi_i 1_{]T_i, T_{i+1}]}(t)$$

where T_is are nonanticipating random times and ϕ_i is $F_{T_i}-$measurable.

Trading strategies are predictable (price processes are not)

Capital gain process:

$$G_t(\phi) = \phi_0 S_0 + \sum_{i=0}^{j-1} \phi_i (S_{T_{i+1}} - S_{T_i}) + \phi_j (S_t - S_{T_j})$$

for $T_j < t \leq T_{j+1}$.

or we can write that as

$$G_t(\phi) = \phi_0 S_0 + \sum_{i=0}^{n} \phi_i (S_{T_{i+1} \wedge t} - S_{T_i} \wedge t)$$

This is called the stochastic integral of ϕ wrt S.

Prop 8.1: If S_t is a martingale then for any simple predictable ϕ the stochastic integral is also a martingale.

Semimartingales: S_t is called a semimartingale if it is nonanticipating, cadlag and the stochastic integral wrt simple predictable processes is continuous (uniform convergence in ω and t implies convergence in probability in the integral)

Every finite variation process is a semimartingale.

Every square integrable martingale is a semimartingale.

BMs, Poisson processes, Levy processes are semimartingales.

Infinite variation deterministic processes, Fractional BMs are not.

Proposition 8.4: S a semimartingale, ϕ a caglad process, π^n a sequence of random partitions of $[0, T]$ then the "Riemann sums" converge in probablity to a process which we call the stochastic integral and denote:

$$\int_0^t \phi_{u-} dS_u$$

It is very important here that in the Riemann sums we compute the integrand at the left endpoint of the interval.

Stochastic integral wrt BM

If ϕ is simple predictable then $\int_0^t dW$ is a martingale and $E(\int_0^t \phi dW) = 0$.

Also,

$$E(|\int_0^T \phi_t dW_t|^2) = E(\int_0^T \phi_t^2 dt)$$

That is called the Isometry Formula.

Using this we can define the stochastic integral for predictable processes by approximating in L^2.

The martingale property is conserved when doing this.

Stochastic Integrals wrt Poisson random measures.

Consider M a random measure on $[0, T]\mathrm{X}R^d$

with intensity $\mu(dt\mathrm{x}dx)$.

Recall that $\tilde{M}_t(A) = M([0,t]\mathrm{x}A) - \mu([0,t]\mathrm{x}A)$ is a martingale.

Also, if $A \cap B = \emptyset$ then $M_t(A)$ and $M_t(B)$ are indep.

Define simple predictable processes:

$$\phi(t,y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{ij} 1_{]T_1,T_{i+1}]}(t) 1_{A_j}(y)$$

where ϕ_{ij} are $\mathcal{F}_{T_i}$-measurable, the sets A_j are disjoint.

The stochastic integral is defined as:

$$\int_0^T \int_{R^d} \phi(t,y) M(dt,dy) = \sum_{i,j=1}^{n,m} \phi_{ij} M(]T_i, T_{i+1}]\mathrm{x}A_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{ij} [M_{T_{i+1}}(A_j) - M_{T_i}(A_j)]$$

similarly as before

$$\int_0^t \int_{R^d} \phi(t,y) M(dt,dy) =$$

$$\sum_{i,j=1}^{n,m} \phi_{ij}[M_{T_{i+1}\wedge t}(A_j) - M_{T_i \wedge t}(A_j)]$$

this process is cadlag and nonanticipating.

We can define the compensated integral

$$\int_0^T \int_{R^d} \phi(t,y) \tilde{M}(dt,dy) = \sum_{i,j=1}^{n,m} \phi_{ij}[M(]T_i, T_{i+1}]\text{x}A_j) -$$

$$\mu(]T_i, T_{i+1}]\text{x}A_j)]$$

Prop 8.7. for simple, predictable processes the compensated integral is a square integrable martingale and verifies

$$E(|\int_0^t \int_{R^d} \phi(t,y) \tilde{M}(dt,dy)|^2) =$$

$$E(\int_0^t \int_{R^d} |\phi(t,y)|^2 \mu(ds,dy))$$

This isometry allows us to extend the compensated integral to square integrable predictable functions (by approx with simple ones).

Example: Suppose that M describes the the jump times and sizes of a a stochastic process S_t:

$$M = J_S(\omega, \cdot) = \sum_{t \in [0,T]}^{\Delta S_t \neq 0} \delta_{t, \Delta S_t}$$

Then:

$$\int_0^T \int_{R^d} \phi(t, y) M(dt, dy) = \sum_{t \in [0,T]}^{\Delta S_t \neq 0} \phi_{t, \Delta S_t}$$

Quadratic variation

Consider the quantity $(X_{t_i+1} - X_{t_i})^2$

It can be rewritten as

$$X_{t_i+1}^2 - X_{t_i}^2 - 2X_{t_i}(X_{t_i+1} - X_{t_i})$$

$$X_T^2 - X_0^2 - 2\sum_{t_i \in \Pi} X_{t_i}(X_{t_i+1} - X_{t_i})$$

If X is a semimartingale we can define the process X_- which is caglad.

Taking the limit (in the partition size) we can define

$$[X, X]_t = |X_T|^2 - 2\int_0^T X_{u-} dX_u$$

It is an increasing process therefore we can define integrals $\int \phi d[X, X]$ path by path.

For any finite variation function it is 0.

$$\Delta[X,X]_t = |\Delta X|^2$$

so, the quadratic variation has continuous paths iff X does.

Martingales have nonzero quadratic variation.

Examples:

Quadratic variation of a Brownian Motion $B = \sigma W$ is $\sigma^2 t$.

For a Poisson Process $[N,N]_t = N_t$

For any finite variation process X:

$$[X,X]_t = \sum_{0\leq s\leq t} |\Delta X_s|^2$$

For a Levy process with triplet (σ, ν, γ)

$$[X, X]_t = \sigma^2 t + \sum_{0 \le s \le t} |\Delta X_s|^2$$

$$\sigma^2 t + \int_{[0,t]} \int_R y^2 J_X(ds, dy)$$

In particular if the process is a symmetric α-stable Levy Process, which has infinite variance, the quadratic variation is well defined.

The quad. variation of a Levy process is again a Levy process, it is a subordinator.

Quadratic covariation

Similarly

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{s-} dY_s - \int_0^t Y_{s-} dX_s$$

If X, Y are semimartingales and ϕ, ψ are predictable, integrable then

$$[\int \phi dX, \int \psi dY]_t = \int_0^t \phi\psi d[X, Y]$$

Itô Formula

For smooth functions:

$$f(g(t) - f(g(0)) = \int_0^t f'(g(s)g'(s)ds$$

When X is a semimartingale:

$$X_t^2 - X_0^2 = 2\int_0^t X_{s-} dX_s + [X, X]_t$$

so there is an extra term

In the case of the Brownian Motion we get:

$$f(X_t) = f(0) + \int_0^t f'(X_s) dW_s +$$

$$\int_0^t \frac{1}{2}\sigma_s^2 f''(X_s)ds$$

What do we do when we add jumps?

For a function x with a finite number of discontinuities we can write:

$$x(t) = \int_0^t b(s)ds + \sum_{i,T_i \le t} \Delta x_i$$

if we apply a function$f$ to x then on the intervals in which x is smooth

$$f(x(T_{i+1}-))-f(x(T_i)) = \int_{T_i}^{T_{i+1}-} f'(x(t))x'(t)dt$$

$$= \int_{T_i}^{T_{i+1}-} f'(x(t))b(t)dt$$

and at the disc points the jump is

$$f(x(T_i))-f(x(T_i-)) = f(x(T_i-)-\Delta x_i)-f(x(T_i-))$$

so for piecewise continuous functions the change of variables becomes:

$$f(x(T)) - f(x(0)) = \int_0^T b(t) f'(x(t-))dt +$$

$$\sum_{i=1}^{n+1} f(x(T_i-) + \Delta x_i) - f(x(T_i-))$$

This is all deterministic.

For a process X we can do this path by path:

$$f(X_T) - f(X_0) = \int_0^T b(t-) f'(X_{t-})dt +$$

$$\sum_{0 \leq t \leq T}^{\Delta X_t \neq 0} f(X_{t-} + \Delta X_t) - f(X_{t-})$$

By considering the jump measure of X:

$$J_X = \sum_{n \geq 1} \delta_{(T_n, \Delta X_{T_n})}$$

we can write the second term as an integral wrt J_X.

We can also compensate the measure (the intensity is $\lambda dt F(dy)$ if X is a compound Poisson process with those parameters) to obtain:

$$\int_0^t \int_R [f(X_s+y)-f(X_s)]\tilde{J}_x(ds,dy)+$$

$$\int_0^t \lambda ds \int_R F(dy)[f(X_s+y)-f(X_s)]$$

the first term is a martingale and the second one is the drift.

Prop 8.18: Itô formula for multidimensional Levy Processes

$$f(t,X_t)-f(0,0)=\int_0^t \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(s,X_{s-})dX_s^i+$$

$$\int_0^t \frac{\partial f}{\partial s}(s,X_s)ds+$$

$$\frac{1}{2}\int_0^t \sum_{i,j=1}^{t} A_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)ds+$$

$$\sum_{0\leq s\leq t}^{\Delta X_s\neq 0}(f(s,X_{s-}+\Delta X_s)-f(s,X_{s-})-$$

$$\sum_{i=1}^{d}\Delta X_s^i\frac{\partial f}{\partial x_i}(s,X_{s-}))$$

Now, if X_t is a Levy process then $Y_t = f(t, X_t)$ is not a Levy process anymore.

But it can be expressed as a in terms of stochastic integrals so it is a semimartingale.

So, it should be helpful to have a version of Itô's formula for semimartingales.

If X is a semimartingale its quadratic variation $[X,X]$ is an increasing process. Then it can be decomposed into a jump part and a continuous part which we will call $[X, X]^c$.

The version for semimartingales replaces the ds in the second derivative term by $d[X, X]^c$.

In Black-Scholes:

$$\frac{dS_t}{S_t} = (\mu + \frac{\sigma^2}{2})dt + \sigma dW_t = dB_t^1$$

or

$$\log(S_t) - \log(S_0) = \mu t + \sigma W_t = B_t^0$$

We can replace B^0 and B^1 by Levy processes