Schonbucher – Chapter 9: Firm Value and Share Priced-Based Models

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(References sited are listed in the book's bibliography, except Miller 1988)

For Intensity and spread-based models of default risk the "focus was on consistent [defaultable bond] pricing, i.e.; finding a not too complicated formal model of the stochastic arrival of a default …"

Structural models are based on modeling a stochastic process for the firm's value; a more fundamental approach for valuing defaultable debt, fundamental in the sense of linking equity pricing and debt pricing. Structural Models and firm value model will be used as synonyms.

Firm value V is assumed to move stochastically. All claims against firm value are modeled as derivative securities, the underlying being firm value.

The Equity in a firm's Capital Structure (the right side of a firm's balance sheet) is a Call Option on firm assets (the left side of a firm's balance sheet). The strike is the par value of firm debt and the expiration date is debt maturity.

Black-Scholes 1973 & Merton 1974: Default occurs or does not occur at debt maturity where V is either not sufficient or is sufficient to payoff debt. The original papers describe a firm value model commonly known as Merton Models. The model is an analogous to a European Call with equity the owner and debt the writer.

Black & Cox 1979 constructed a Structural Model where default can occur prior to debt maturity. Default prior to debt maturity is not allowed in the Merton Structural Model. To account for pre maturity default, default is triggered when V touches a barrier $\overline{K}$. Such an options model is known as a knock out option.

Black-Scholes, Merton Construct – The European Call Option:

Balance Sheet Accounting Equality:

$$V = S + \overline{B}$$

Firm value V is a geometric Brownian motion.

$$dV = \mu V dt + \sigma V dW$$

With:

V : firm value.
$\overline{B}(V,t)$: total market value of defaultable debt
$S(V,t)$: total value of outstanding equity (the firm's market capitalization i.e. share price times total outstanding shares).
$\overline{D}$: face value that is par value of defaultable debt.
$\overline{S}$: total number of outstanding equity shares.
r : risk free interest rate and assumed constant.

Payoff of a European Call Option at expiration and therefore the payoff of debt and equity at debt maturity is

Total Debt: $\overline{B}(V,t) = \min(\overline{D}, V)$
Total Equity: $S(V,t) = \max(V - \overline{D}, 0)$

which is the payoff of a call option for the option writer (debt) and buyer (equity).

To show the payoffs are options that satisfy the B-S PDE, we set up a portfolio of a unit of bonds and Δ shares

$$\Pi = \overline{B}(V,t) + \Delta S(V,t)$$
$$d\Pi = d\overline{B} + \Delta dS$$

We apply Ito's lemma

$$d\Pi = \left(\frac{\partial \overline{B}}{\partial t} + \frac{1}{2}\frac{\partial^2 \overline{B}}{\partial V^2} + \Delta\frac{\partial S'}{\partial t} + \frac{1}{2}\Delta\frac{\partial^2 S'}{\partial V^2}\right)dt + \left(\frac{\partial \overline{B}}{\partial V} + \Delta\frac{\partial S}{\partial V}\right)dW$$

and we see the hedge ratio of debt to equity that creates a riskless portfolio of debt and equity, that is eliminates dW .

$$\Delta = -\frac{\partial \overline{B}}{\partial V}\Big/\frac{\partial S}{\partial V}$$

The result is the B-S PDE:

$$F_t + \frac{1}{2}\sigma^2 V^2 F_{VV} + rVF_V - rF = 0$$

With boundary conditions and integrating the solution for the equity value is

$$S(V,t) = C^{BS}(V,t,T,\overline{D},\sigma,r)$$

Where C^{BS} is a Black-Scholes European call option price, t is today and T is debt maturity date.

Schonbucher Comments on the Merton Model

Schonbucher states while V is not tradable, S and $\overline{B}$ are tradable and can synthesize V validating the B-S PDE arbitrage (personal comment, this assertion needs careful consideration).

<u>Model Results</u>: 1) Equity and debt are priced and priced consistently. 2) Equity and debt are hedges of each other.

<u>Model Problems</u>: 1) V and σ are not observable, 2) Default only occurs at debt maturity.

[The relationship of equity and debt as options can also be seen in put/call parity

$$Underlying = Call - Put + (Strike\,\Pr ice)e^{-r(T-t)}$$

$$V = S + \overline{B}$$
$$\overline{B} = \overline{D}e^{-r(T-t)} - x$$
$$V = S - x + \overline{D}e^{-r(T-t)}$$

With x the difference in price of two bonds exactly the same except one is defaultable obligation of the firm and the other not defaultable. With the put call parity analogy, equity is a long call option and debt is a a risk free rate lender and a short put option with x the put price.]

Black & Cox Knock-Out Option Construct – Dealing with Default Prior to Debt Maturity:

To introduce defaults prior to debt maturity, Black & Cox described a covenant based model that is a knock out option.

Assume: $dV = rVdt + \sigma VdW$ geometric Brownian motion under the martingale measure Q.

Assume debt has safety covenants allowing creditors to liquidate the firm eliminating the call option when $V \le K(t)$ where $K(t)$ is the value specified in the covenant. Default occurs at τ

$$\tau = \inf\{t \ge 0 \,|\, V(t) \le K(t)\}$$

With the payoff being:

$\overline{B}(V, t = T) = \min\{\overline{D}, V\}$ and $S(V, t = T) = (V - \overline{D})^{+}$ if no default or
$\overline{B}(V = K(t), t) = V - c = K(t) - c$ and $S(V = K(t), t) = 0$ if default is triggered at some time between inclusively t and T. c are bankruptcy costs.

Without bankruptcy costs c with a covenant $\overline{D} \le K(t)$ greater then the debt face value, creditors would not suffer a loss in bankruptcy since they would liquidate the firm the moment they started experiencing losses.

Model for default is the first hitting time τ of a diffusion process at a fixed barrier. $P(T,W)$ is the probability of not hitting the barrier. Given Ito's lemma and zero drift, the expectation for $P(T,W)$ is

$0 = P_t + \frac{1}{2}P_{WW}$ with a solution $P(t,W) = N(\frac{x}{\sqrt{t}}) - N(-\frac{x}{\sqrt{t}})$; the credit spreads plotted from this solution on page 262 figure 9.3.

Significant Problem with All Structural Models:

In Structural Models the probability of a default in a short time interval is smaller then than of order Δt. The model therefore predicts and prices a zero spread at short maturities. Empirically this prediction is not observed and is a significant issue with the model.

Practical Implementation: KMV

The <u>default point</u>: "asset value at which the firm will default"; KMV places that point somewhere between the face amount of short and long term liabilities. Similar to a barrier option.

A time horizon used to calculate survival probabilities: one to five years or more.

Asset value (firm value) cannot be determined from the balance sheet due to off balance sheet obligations (such as pension funds).

Asset value V and asset volatility σ_V are inferred (solved) from observed total equity market capitalization and historical equity volatility.

To find V & σ_V:

$S^{'}$ is the observed market value of equity. S is a function from the Merton Model:

Analytically we have from the Merton Model

$$S = C^{BS}(V, \sigma_V; r, T, \overline{D})$$

Applying Ito's lemma

$$dS = (...)dt + C_V^{BS}(V, \sigma_V, ...)V\sigma_V dW$$

where C_V^{BS} is the 1st derivative of the call option C^{BS} with respect to V.

Empirically we observe $\frac{dS'}{S'} = \sigma_{S'} dW$

Therefore equating dW : $\frac{dS'}{S'\sigma_{S'}} = \frac{dS - (...)dt}{C_V^{BS}V\sigma_V}$

$$\frac{dS'}{S'\sigma_{S'}} = \frac{dS - (...)dt}{C_V^{BS}V\sigma_V} \rightarrow \frac{dS'}{S'\sigma_{S'}} = \frac{dS}{C_V^{BS}V\sigma_V} \rightarrow S'\sigma_{S'} = C_V^{BS}V\sigma_V$$

"We can view $S'\sigma_{S'} = C_V^{BS}V\sigma$ as a first approximation and not an exact relationship, but it will give largely realistic values." Page 276

We now have two equations and two unknowns: V, σ_V

$$S' = C^{BS}(V, \sigma_V; r, T, \overline{D})$$
$$S'\sigma_{S'} = C_V^{BS}V\sigma_V$$

with observable values on the left side and two unknowns are the right V and σ_V whose values we can now calculate.

KMV - Distance to Default

[Distance to Default] =

{ log(Market Value of Assets (V)] – log[Default Point] } / [Asset Volatility (σ_V)] }

If V is Gaussian then an expected default frequency (EDF) for the time horizon is determined by the distance to default. (While not specified, distance to default must be in the Real/Actual Measure (not risk neutral). If V is not Gaussian then a historical distribution is associated with the Gaussian distance to default (some functional relationship) to determine the reported EDF. The exact procedure for determining the EDF is proprietary to Moody's/KMV.

[From: MODELING DEFAULT RISK – MODELINGMETHODOLOGY AUTHORS: Peter Crosbie & Jeff Bohn

> There are essentially three steps in the determination of the default probability of a firm:
>
> Estimate asset value and volatility: In this step the asset value and asset volatility of the firm is estimated from the market value and volatility of equity and the book value of liabilities.
>
> Calculate the distance-to-default: The distance-to-default (DD) is calculated from the asset value and asset volatility (estimated in the first step) and the book value of liabilities.
>
> Calculate the default probability: The default probability is determined directly from the distance-to-default and the default rate for given levels of distance-to-default.
>
> If the market price of equity is available, the market value and volatility of assets can be determined directly using an options pricing based approach, which recognizes equity as a call option on the underlying assets of the firm.

We observe the market value of the equity and solve backwards for the market value of Assets.

…we solve the following two relationships simultaneously:

$$\begin{bmatrix} \text{Equity} \\ \text{Value} \end{bmatrix} = OptionFunction\left(\begin{bmatrix} \text{Asset} \\ \text{Value} \end{bmatrix}, \begin{bmatrix} \text{Asset} \\ \text{Volatility} \end{bmatrix}, \begin{bmatrix} \text{Capital} \\ \text{Structure} \end{bmatrix}, \begin{bmatrix} \text{Interest} \\ \text{Rate} \end{bmatrix} \right)$$

$$\begin{bmatrix} \text{Equity} \\ \text{Volatility} \end{bmatrix} = OptionFunction\left(\begin{bmatrix} \text{Asset} \\ \text{Value} \end{bmatrix}, \begin{bmatrix} \text{Asset} \\ \text{Volatility} \end{bmatrix}, \begin{bmatrix} \text{Capital} \\ \text{Structure} \end{bmatrix}, \begin{bmatrix} \text{Interest} \\ \text{Rate} \end{bmatrix} \right)$$

Asset value and volatility are the only unknown quantities in these relationships and thus the two equations can be solved to determine the values implied by the current equity value, volatility and capital structure.

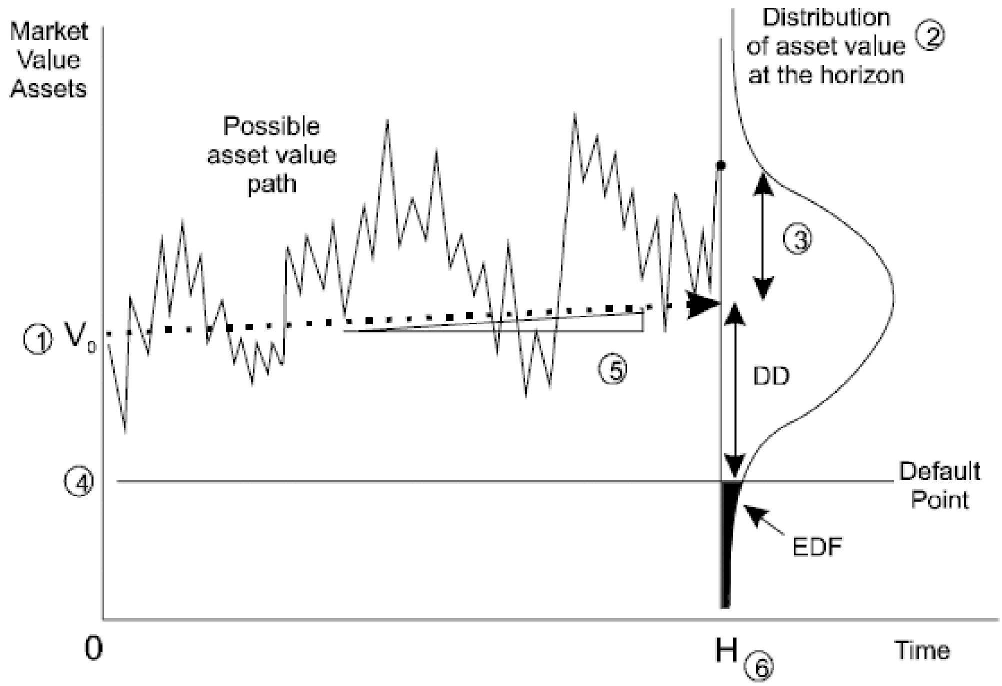

FIGURE 8

1. The current asset value.
2. The distribution of the asset value at time H .
3. The volatility of the future assets value at time H .
4. The level of the default point, the book value of the liabilities.
5. The expected rate of growth in the asset value over the horizon.
6. The length of the horizon, H .

The first four variables, asset value, future asset distribution, asset volatility and the level of the default point, are the really critical variables. The expected growth in the asset value has little default discriminating power and the analyst defines the length of the horizon. If the value of the assets falls below the default point, then the firm defaults. Therefore, the probability of default is the probability that the asset value will fall below the default point. This is the shaded area (EDF value) below the default point in Figure 8.

$$\begin{bmatrix}\text{Distance}\\ \text{to Default}\end{bmatrix} = \frac{\begin{bmatrix}\text{Market Value}\\ \text{of Assets}\end{bmatrix} - \begin{bmatrix}\text{Default}\\ \text{Point}\end{bmatrix}}{\begin{bmatrix}\text{Market Value}\\ \text{of Assets}\end{bmatrix}\begin{bmatrix}\text{Asset}\\ \text{Volatility}\end{bmatrix}}$$

Formula for calculating Distance to Default.

Why isn't information from the bond or credit derivatives market included?

There is a whole class of models, usually called reduced-form models that relate credit spreads and default probabilities. Our research indicates these models can be difficult to parameterize given the noise in debt market data. There is nothing wrong with these models per se, indeed in theory they hold the promise of some advantages over the causal model described in this paper. However, the empirical stability of the structural models combined with their economic interpretation makes them more effective for default probability estimation. Reduced-form models also suffer from the need to make more assumptions about the relationship of default probability and loss given default in order to arrive at useful results. As credit markets become more liquid and the data more accurately reported, reduced-form models will better complement the analysis facilitated with structural models.]

Implementing Barrier Default models (Black & Cox):

Since there is a finite distance to the barrier, a continuous process cannot reach it in the next instance. Therefore for short maturities Structural Models predict a zero credit spread, which is empirically incorrect. How then are zero short-term spreads eliminated that is how do zero prices for short term credit avoided?

1) Introduce jumps in the process for V . (Zhou 2001)

2) Uncertainty is associated with the exact value of $V(t)$ or the barrier $\overline{K}(t)$. (Duffie & Lando 2001, Giesecke 2001, Finger et al. 2002).

Empirical Evidence:

Structural models predict a humped shape for the term structure of credit. The hump is observed with firms of bad credits, not observed with firms of investment grade credit.

Structural model's ordinal ranking of firm riskiness does not match market observed ranking. (Lardic & Rouzeau 1999).

Eom et al. (2000) cross section study across corporate bonds - Structural models in various constructs:

1) Pricing errors are numerous,
2) Significant underestimating of spreads (less risky the market valuations),
3) A version of the barrier option overestimated spreads.

"… all models had problems for short maturities or high quality, and the predictive power of the models was very poor in all cases." Page 285.

Schonbucher's Comments

The models' strong point is the focus on fundamentals. "[The empirical studies} only show that there is al lot of work yet to be done until a proper, reliable and robust [firm value] model is found which somehow describes the link between equity and debt prices which *must* be there – somewhere." Page 285.

The weak points – requires unobservable inputs and is too complex to analyze in the real world, "if one were to model the full set of claims on the value of the assets of a medium-sized corporation one may very well have to price 20 or more classes of claims…" page 286.

Personal Comments:

1) The intensity models are well suited for security dealers and commercial banks since neither positions equity.
2) Pricing of debt *and* equity in the Structural Model result from economic equilibrium. The firm value can be described in terms of microeconomic capital theory. Options arbitrage transmits the economic firm value into debt and equity pricing. Grounded in economic theory enables using the tools of financial economics to manage debt.
3) The pricing of equity and debt is analytical, not ad hoc. Value can be anticipated which then tested empirically. Ad hoc methodology requires empirical observation initially to then see presumed associations.
4) Equity and debt as the owner and writer of a call option is odd. The model implies prior to debt maturity, debt is the owner of the firm not equity. And if there is no debt with equity a call option there is no firm owner.
5) A subtlety of equity as a call is the borrowing. Borrowing and lending in traditional calls is at a rate independent of the underlying. The borrowing in the Merton Model has optionality associated with it and therefore not independent of the underlying.
6) Merton Miller's put call parity analogy where equity is a long call and debt is a combination of a short put and risk free lending begs the question as to who is the call writer and put owner.

I am currently writing a paper to address these issues.

Bibliography:

Miller, Merton H., "The Modigliani-Miller Propositions After Thirty Years", *The Journal of Economic Perspectives*, Vol. 2, No. 4 (Autumn, 1988), Page 110.

Bohn, Jeff, Peter Crosbie, "Default Risk: Modeling Methodology", © 2005 Moody's KMV Company. December 18, 2003.