SOLUTIONS TO PRACTICE TEST 1

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1. In the xy-plane, the curve with parametric equations $x = \cos t$ and $y = \sin t$, $0 \le t \le \pi$, has length

(A) 3
(B) π
(C) 3π
(D) $3/2$
(E) $\pi/2$

Solution: The velocity is given by $\dot{x} = -\sin t$ and $\dot{y} = \cos t$. The speed is given by $\left[(-\sin t)^2 + (\cos t)^2\right]^{1/2} = [1]^{1/2} = 1$. The length of the curve is then $\displaystyle\int_0^\pi 1\,dt = \pi$. Answer: (B) $\square$

Alternate solution: The curve is a semicircle of radius 1, so its length is π. Answer: (B) $\square$

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2. Which of the following is an equation of the line tangent to the graph of $y = x + e^x$ at $x = 0$.

(A) $y = x$
(B) $y = x + 1$
(C) $y = x + 2$
(D) $y = 2x$
(E) $y = 2x + 1$

Solution: The slope is $[(d/dx)(x + e^x)]_{x:\to 0} = [1 + e^x]_{x:\to 0} = 1 + e^0 = 2$. The y-coordinate of the point of tangency is $[x + e^x]_{x:\to 0} = 0 + e^0 = 1$, so the point of tangency is $(0, 1)$. An equation of the line is therefore $y - 1 = 2(x - 0)$, or $y = 2x + 1$. Answer: (E) $\square$

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3. If V and W are 2-dimensional subspaces of $\mathbb{R}^4$, what are the possible dimensions of $V \cap W$?

(A) 1 only
(B) 2 only
(C) 0 and 1 only
(D) 0, 1 and 2 only
(E) 0, 1, 2, 3 and 4

Solution: The minimum possible dimension of $V + W$ is

$$\max\{\dim V, \dim W\} \quad = \quad \max\{2,2\} \quad = \quad 2.$$

The maximum possible dimension of $V + W$ is

$$\min\{\,(\dim V) + (\dim W)\,,\, \dim \mathbb{R}^4\,\} \quad = \quad \min\{2+2,4\} \quad = \quad 4.$$

So the set of possible dimensions of $V + W$ is $\{2,3,4\}$. As

$$(\dim V) \ + \ (\dim W) \quad = \quad (\dim (V + W)) \ + \ (\dim (V \cap W)),$$

we see that $4 = 2 + 2 = (\dim (V + W)) + (\dim (V \cap W))$, so

$$\dim (V \cap W) \quad = \quad 4 \ - \ (\dim (V + W)).$$

Then the set of possible dimensions of $V \cap W$ is

$$\{\,4-2\,,\,4-3\,,\,4-4\,\} \quad = \quad \{2,1,0\}.$$

Answer: (D) $\square$

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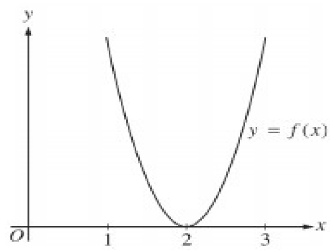
4. Let k be the number of real solutions of the equation $e^x + x - 2 = 0$ in the interval $[0,1]$, and let n be the number of real solutions that are <u>not</u> in $[0,1]$. Which of the following is true?

(A) $k = 0$ and $n = 1$
(B) $k = 1$ and $n = 0$
(C) $k = n = 1$
(D) $k > 1$
(E) $n > 1$

Solution: For all $x \in \mathbb{R}$, we have $[d/dx][e^x + x - 2] = e^x + 1 > 0$, so $e^x + x - 2$ is increasing in x. So, since $[e^x + x - 2]_{x:\to 0} = 1 + 0 - 2 < 0$ and $[e^x + x - 2]_{x:\to 0} = e + 1 - 2 > 0$, it follows that the equation $e^x + x - 2 = 0$ has one solution in $[0,1]$, and no solutions that are not in $[0,1]$. That is, $k = 1$ and $n = 0$. Answer: (B) $\square$



5. Suppose b is a real number and $f(x) = 3x^2 + bx + 12$ defines a function on the real line, part of which is graphed above. Then $f(5) =$

(A) 15
(B) 27
(C) 67
(D) 72
(E) 87

Solution: We have $f'(x) = 6x + b$, so $f'(2) = 12 + b$. From the graph, $f'(2) = 0$. Thus $12 + b = 0$, so $b = -12$. Then $f(x) = 3x^2 - 12x + 12$, so $f(5) = 3 \cdot 5^2 - 12 \cdot 5 + 12 = 75 - 60 + 12 = 27$. Answer: (B) $\square$

Alternate solution: We have $f(2) = 3 \cdot 2^2 + b \cdot 2 + 12$. From the graph, $f(2) = 0$. Thus $12 + 2b + 12 = 0$, so $2b = -24$, so $b = -12$. Then $f(x) = 3x^2 - 12x + 12$, so $f(5) = 3 \cdot 5^2 - 12 \cdot 5 + 12 = 75 - 60 + 12 = 27$. Answer: (B) $\square$

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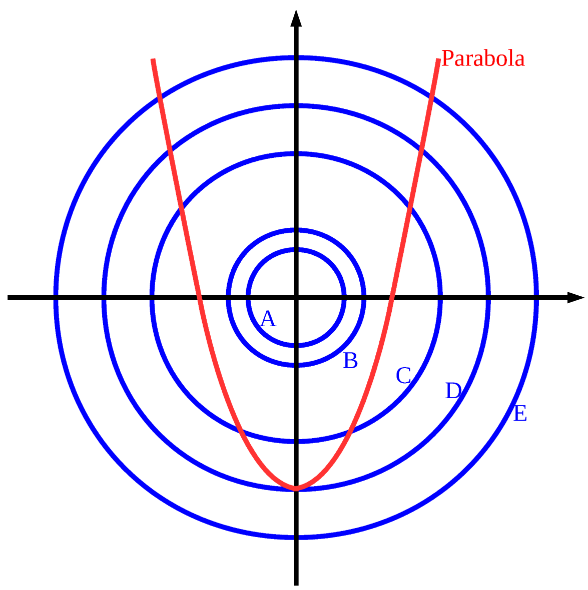
6. Which of the following circles has the greatest number of points of intersection with the parabola $x^2 = y + 4$.

(A) $x^2 + y^2 = 1$
(B) $x^2 + y^2 = 2$
(C) $x^2 + y^2 = 9$
(D) $x^2 + y^2 = 16$
(E) $x^2 + y^2 = 25$

Solution:



The parabola in question is the graph of $y = x^2 - 4$, and is obtained by moving $y = x^2$ down four units. Its x-intercepts are -2 and 2, and its y-intercept is -4. It is shown in the figure above in red, marked "Parabola". The circles in answers (A),...,(E) are shown in blue, marked "A",...,"E".

The circles A and B have 0 intersection points. The circle C has 4 intersection points. The circle D has 3 intersection points. The circle E has 2 intersection points. Answer: (C) □

Alternate solution: We will count the intersections in (C). The other counts are similar, but as we move from (C) to the other parts, the new versions of a and b, defined below, will be different. Since, below, we need to compute $\sqrt{a+4}$

Let $a := (-1-\sqrt{1+20})/2$ and $b := (-1+\sqrt{1+20})/2$. Then $a+4>0$ and $b+4>0$. For all $x,y \in \mathbb{R}$,

$$\begin{array}{ll} [\ (\ x^2+y^2=9\ )\text{ and }(\ x^2=y+4\ )\ ] & \text{iff} \\ [\ (\ y+4+y^2=9\ )\text{ and }(\ x^2=y+4\ )\ ] & \text{iff} \\ [\ (\ y^2+y-5=0\ )\text{ and }(\ x^2=y+4\ )\ ] & \text{iff} \\ [\ (\ (\ y=a\ )\text{ or }(\ y=b\ )\ )\text{ and }(\ x^2=y+4\ )\ ] & \text{iff} \end{array}$$

$[(y=a)$ and $(x \in \{\pm\sqrt{a+4}\})]$ or $[(y=b)$ and $(x \in \{\pm\sqrt{b+4}\})]$. There are therefore four intersection points:

$$(\sqrt{a+4},a), \qquad (-\sqrt{a+4},a), \qquad (\sqrt{b+4},b), \qquad (-\sqrt{b+4},b)$$

Now, if you know "Bézout's Theorem", you know that, over the real numbers, the intersection of two distinct quadratics (like $x^2+y^2=9$ and $x^2=y+4$) cannot have more than four points. This shows that the intersection counts for (A), (B), (D) and (E) must all be ≤ 4. So either the problem has two answers, or the only correct answer is (C). If you **don't** know Bézout's Theorem, or don't appreciate the kind of thinking in this paragraph, you can modify the intersection count for (C) and get the counts for (A), (B), (D) and (E).

As we move from (C) to other parts of the problem, a and b will change, and, for displaying intersection points, we need to take the square root of $a+4$ and $b+4$. So much depends on whether $a+4$ is positive, negative or zero and, also, much depends on whether $b+4$ is positive, negative or zero. We leave it to the reader to do show that (A) and (B) have no intersection points. We leave it to the reader to do show that (D) has three intersection points. We leave it to the reader to do show that (E) has two intersection points.

Alternate solution: Let c be a real number. We seek to count intersections of $x^2+y^2=c$ with the parabola $x^2=y+4$. To solve (A), we'll set $c=1$. To solve (B), we'll set $c=2$. To solve (C), we'll set $c=9$. To solve (D), we'll set $c=16$. To solve (E), we'll set $c=25$.

For all $x,y \in \mathbb{R}$,

$$\begin{array}{ll} [\ (\ x^2+y^2=c\ )\text{ and }(\ x^2=y+4\ )\ ] & \text{iff} \\ [\ (\ y+4+y^2=c\ )\text{ and }(\ x^2=y+4\ )\ ] & \text{iff} \\ [\ (\ y^2+y-(c-4)=0\ )\text{ and }(\ x^2=y+4\ )\ ]. & \end{array}$$

Let $\Delta := 1+4(c-4)$. If $\Delta<0$, then

$$[\ (\ x^2+y^2=c\ )\text{ and }(\ x^2=y+4\ )\ ]$$

has no real solutions; that is, there are no intersection points. In (A), we have $c = 1$, so $\Delta = 1 + 4 \cdot (-3) < 0$. In (B), we have $c = 2$, so $\Delta = 1 + 4 \cdot (-2) < 0$. Thus (A) and (B) have no intersection points.

It turns out that $\Delta = 0$ does not occur in (A),...,(E). So assume, for the remainder, that $\Delta > 0$. Define $a := (-1 - \sqrt{\Delta})/2$ and define $b := (-1 + \sqrt{\Delta})/2$. Note that $a < b$. For all $x, y \in \mathbb{R}$,

$$\begin{array}{ll} [\ (\ y^2 + y - (c-4) = 0\ )\ \text{and}\ (\ x^2 = y + 4\ )\ ] & \text{iff} \\ [\ (\ (\ y = a\ )\ \text{or}\ (\ y = b\ )\ )\ \text{and}\ (\ x^2 = y + 4\ )\ ]. & \end{array}$$

If $a + 4 = 0$, then $b + 4 > 0$, and, in this case, we have three intersection points:

$$(0, a), \qquad (\sqrt{b+4}, b), \qquad (-\sqrt{b+4}, b).$$

In (D), we have $c = 16$, so $\Delta = 1 + 4 \cdot 12 = 49$. Then

$$a \quad = \quad (-1 - 7)\,/\,2 \quad = \quad -4,$$

so $a + 4 = 0$. So, for (D), we have three intersection points.

If $a + 4 < 0$ and $b + 4 > 0$, then we have two intersection points:

$$(\sqrt{b+4}, b), \qquad (-\sqrt{b+4}, b).$$

In (E), we have $c = 25$, so $\Delta = 1 + 4 \cdot 21 = 85$. Then

$$a \quad = \quad (-1 - \sqrt{85})\,/\,2 \quad < \quad (-1 - \sqrt{49})\,/\,2 \quad = \quad -4$$

and $b = (-1 + \sqrt{85})/2 > -4$, so $a + 4 < 0$ and $b + 4 > 0$. So, for (E), we have two intersection points.

If $a + 4 > 0$, then $b + 4 > 0$, then we have four intersection points:

$$(\sqrt{a+4}, a), \qquad (-\sqrt{a+4}, a), \qquad (\sqrt{b+4}, b), \qquad (-\sqrt{b+4}, b).$$

In (C), we have $c = 9$, so $\Delta = 1 + 4 \cdot 5 = 21$. Then

$$a \quad = \quad (-1 - \sqrt{21})\,/\,2 \quad > \quad (-1 - \sqrt{49})\,/\,2 \quad = \quad -4.$$

So, for (C), we have four intersection points. Answer: (C) $\square$

7. $\displaystyle\int_{-3}^{3} |x+1|\,dx =$
(A) 0
(B) 5
(C) 10
(D) 15
(E) 20

Solution: The integral is equal to

$$\begin{aligned} & \left[\int_{-3}^{-1} |x+1|\,dx\right] + \left[\int_{-1}^{3} |x+1|\,dx\right] \\ = \; & \left[\int_{-3}^{-1} (-x-1)\,dx\right] + \left[\int_{-1}^{3} (x+1)\,dx\right] \\ = \; & \left[-\frac{x^2}{2} - x\right]_{x:\to -3}^{x:\to -1} + \left[\frac{x^2}{2} + x\right]_{x:\to -1}^{x:\to 3} \\ = \; & \left[-\frac{1-9}{2} - 2\right] + \left[\frac{9-1}{2} + 4\right] \\ = \; & -(-4) - 2 + 4 + 4 \quad = \quad 10. \end{aligned}$$

Answer: (C) □

Alternate solution: Between $x = -3$ and $x = 3$, the region below the graph of $y = |x+1|$ breaks up into two right isosceles triangles. The leftmost triangle has legs of length 2, and therefore has area $(1/2)(2)(2) = 2$. The rightmost triangle has legs of length 4, and therefore has area $(1/2)(4)(4) = 8$. Total: $2+8 = 10$. Answer: (C) □

8. What is the greatest possible area of a triangular region with one vertex at the center of a circle of radius 1 and the other two vertices on the circle?

(A) $1/2$
(B) 1
(C) $\sqrt{2}$
(D) π
(E) $(1+\sqrt{2})/4$

Solution: Such a triangle is always isosceles with two sides of length 1. It therefore can be split into two congruent right triangles, both of hypotenuse 1. If x is the length of one of the legs of one of the two right triangles, then $0 < x < 1$ and, moreover, $\sqrt{1-x^2}$ is the length of the other leg of that triangle. Thus the area inside each right triangle is $(1/2)x\sqrt{1-x^2}$, so the area inside the isosceles triangle is $x\sqrt{1-x^2}$.

We wish to maximize $f(x) = x\sqrt{1-x^2}$ on $0 < x < 1$. We have

$$\begin{aligned} f'(x) &= \left[\sqrt{1-x^2}\right] + x\left[\frac{-2x}{2\sqrt{1-x^2}}\right] \\ &= \left[\frac{1-x^2}{\sqrt{1-x^2}}\right] + \left[\frac{-x^2}{\sqrt{1-x^2}}\right] \\ &= \frac{1-2x^2}{\sqrt{1-x^2}}. \end{aligned}$$

Then $f'(x) > 0$ on $0 < x < 1/\sqrt{2}$. Also, $f'(x) < 0$ on $1/\sqrt{2} < x < 1$. Thus $f(x)$ is increasing on $0 < x < 1/\sqrt{2}$. Also, $f(x)$ is decreasing on $1/\sqrt{2} < x < 1$. Then $f(x)$ attains a global maximum at $x = 1/\sqrt{2}$. So, since $f(x) = x\sqrt{1-x^2}$, the maximum value is

$$f\left(\frac{1}{\sqrt{2}}\right) = \left[\frac{1}{\sqrt{2}}\right]\left[\sqrt{1-\frac{1}{2}}\right] = \left[\frac{1}{\sqrt{2}}\right]\left[\frac{1}{\sqrt{2}}\right] = \frac{1}{2}.$$

Answer: (A) $\square$

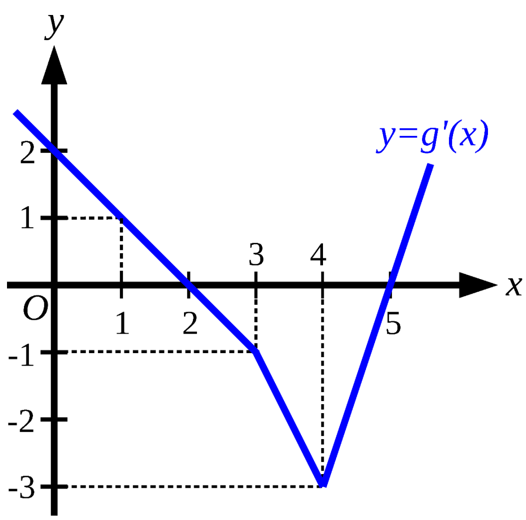
$$\begin{aligned} J &= \int_0^1 \sqrt{1-x^4}\,dx \\ K &= \int_0^1 \sqrt{1+x^4}\,dx \\ L &= \int_0^1 \sqrt{1-x^8}\,dx \end{aligned}$$

9. Which of the following is true for the definite integral shown above?
   (A) $J < L < 1 < K$
   (B) $J < L < K < 1$
   (C) $L < J < 1 < K$
   (D) $L < J < K < 1$
   (E) $L < 1 < J < K$

Solution: On $0 < x < 1$, we have:

$$\begin{array}{lccccccccc} & -1 & < & -x^4 & < & -x^8 & < & 0 & < & x^4, \\ \text{which implies} & 0 & < & 1-x^4 & < & 1-x^8 & < & 1 & < & 1+x^4, \\ \text{which implies} & & & \sqrt{1-x^4} & < & \sqrt{1-x^8} & < & 1 & < & \sqrt{1+x^4}. \end{array}$$

Thus $J < L < 1 < K$. Answer: (A) □



10. Let g be a function whose derivative g' is continuous and has the graph shown above. Which of the following values of g is the largest?

(A) $g(1)$
(B) $g(2)$
(C) $g(3)$
(D) $g(4)$
(E) $g(5)$

Solution: Because $g' > 0$ on $(0, 2)$, and because g is continuous on $[0, 2]$, it follows that g is increasing on $[0, 2]$. Because $g' < 0$ on $(2, 5)$, and because g is continuous on $[2, 5]$, it follows that g is decreasing on $[2, 5]$. Then, on $[0, 5]$, g attains its maximum value at 2. Answer: (B) $\square$

11. Of the following, which is the best approximation of

$$\left[\sqrt{1.5}\,\right]\left[(266)^{3/2}\right]?$$

(A) $1,000$
(B) $2,700$
(C) $3,200$
(D) $4,100$
(E) $5,300$

Solution: We have:

$$\begin{aligned}
\left[\sqrt{1.5}\right]\left[(266)^{3/2}\right] &= \left[\sqrt{1.5}\right]\left[\sqrt{(266)^3}\right] \\
&= \left[\sqrt{1.5}\right]\left[\sqrt{266}\right]\left[\sqrt{(266)^2}\right] \\
&= \left[\sqrt{(1.5)(266)}\right][266] \\
&= \left[\sqrt{266+133}\right][266] \\
&= \left[\sqrt{399}\right][266] \\
&\approx< \left[\sqrt{400}\right][266] \\
&= [20]\,[266] \\
&= 5,320
\end{aligned}$$

Answer: (E) $\square$

12. Let A be a 2×2 matrix for which there is a constant k such that the sum of the entries in each row and each column is k. Which of the following must be an eigenvector of A?

I. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

II. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

III. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I, II and III

Solution: Choose a, b, c, d such that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \text{and} \qquad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

In particular, if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which shows that I and II are not necessarily eigenvectors.

On the other hand, for any A, we have

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which shows that III is necessarily an eigenvector. Answer: (C) $\square$

13. A total of x feet of fencing is to form three sides of a level rectangular yard. What is the maximum possible area of the yard, in terms of x?

(A) $x^2/9$
(B) $x^2/8$
(C) $x^2/4$
(D) x^2
(E) $2x^2$

Solution: Along the three fenced sides, let the side lengths be t feet, $x-2t$ feet and t feet. We then wish to maximize the area

$$A(t) \;=\; (x-2t)t \qquad \text{on} \qquad 0 \le t \le x/2.$$

Since the graph of A is a concave down parabola, and since

$$A(0) \quad = \quad A(x/2) \quad = \quad 0,$$

it follows that $A(t)$ attains its maximum value

$$\text{halfway between} \qquad t=0 \qquad \text{and} \qquad t=x/2,$$

i.e., at $t=(0+(x/2))/2=x/4$. The maximum value is

$$A(x/4) \;=\; (x-2(x/4))(x/4) \;=\; (x/2)(x/4) \;=\; x^2/8.$$

Answer: (B) □

Alternate Solution: Along the three fenced sides, let the side lengths be t feet, $x-2t$ feet and t feet. We then wish to maximize the area

$$A(t) \;=\; (x-2t)t \qquad \text{on} \qquad 0 \le t \le x/2.$$

Then $A'(t)=-2t+x-2t=x-4t$. Since the graph of A is a concave down parabola, and since

$$[\ A'(t) \;=\; 0\ ] \qquad \Leftrightarrow \qquad [\ t=x/4\ ],$$

we see that $A(t)$ attains its maximum at $t=x/4$, with maximum value $A(x/4)=(x-(x/2))(x/4)=(x/2)(x/4)=x^2/8$. Answer: (B) □

Alternate Solution: Along the three fenced sides, let the side lengths be t feet, $x-2t$ feet and t feet. We then wish to maximize the area

$$A(t) \;=\; (x-2t)t \qquad \text{on} \qquad 0 \le t \le x/2.$$

By the arithmetic-geometric mean inequality, for all $a, b \geq 0$, we have: $\sqrt{ab} \leq (a+b)/2$. Equality happens when $a = b$. Replacing a by $2t$ and b by $x - 2t$, we see that $\sqrt{2t(x-2t)} \leq x/2$. Squaring this, and then dividing by 2, we get $t(x-2t) \leq x^2/8$. That is, $A(t) \leq x^2/8$. Equality happens when $2t = x - 2t$, *i.e.*, when $t = x/4$. The maximum value is then $x^2/8$. Answer: (B) $\square$

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14. What is the units digit in the standard decimal expansion of the number 7^{25}?

(A) 1
(B) 3
(C) 5
(D) 7
(E) 9

Solution: In this solution, congruences ($\equiv$) are all mod 10. We have

$$\begin{aligned} 7^2 = 7 \cdot 7 = 49 &\equiv 9, \qquad \text{so} \\ 7^3 = 7 \cdot 7^2 \equiv 7 \cdot 9 = 63 &\equiv 3, \qquad \text{so} \\ 7^4 = 7 \cdot 7^3 \equiv 7 \cdot 3 = 21 &\equiv 1. \end{aligned}$$

Then $7^{25} = 7 \cdot (7^4)^6 \equiv 7 \cdot 1^6 = 7$. Answer: (D) $\square$

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15. Let f be a continuous real-valued function defined on the closed interval $[-2, 3]$. Which of the following is NOT necessarily true?

(A) f is bounded.
(B) $\displaystyle\int_{-2}^{3} f(t)\, dt$ exists.
(C) For each c between $f(-2)$ and $f(3)$, there is an $x \in [-2, 3]$ such that $f(x) = c$.
(D) There is an M in $f([-2, 3])$ such that $\displaystyle\int_{-2}^{3} f(t)\, dt = 5M$.
(E) $\displaystyle\lim_{h \to 0} \frac{[f(h)] - [f(0)]}{h}$ exists.

Solution: By the Extreme Value Theorem, (A) is necessarily true. By the construction of integration, (B) is necessarily true. By the Intermediate Value Theorem, (C) is necessarily true. By the Integral Mean Value Theorem, (D) is necessarily true. Finally, (E) is NOT necessarily true, *e.g.*, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) \quad = \quad \begin{cases} x \cdot (\sin(1/x)), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Answer: (E) $\square$

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16. What is the volume of the solid formed by revolving, about the x-axis, the region in the first quadrant of the xy-plane bounded by: the coordinate axes and the graph of the equation $y = \dfrac{1}{\sqrt{1+x^2}}$?

(A) $\pi/2$
(B) π
(C) $\pi^2/4$
(D) $\pi^2/2$
(E) ∞

Solution: By the disk method, the volume is

$$\begin{aligned} \int_0^\infty \pi \left[\frac{1}{\sqrt{1+x^2}} \right]^2 dx \quad &= \quad \pi \int_0^\infty \frac{dx}{1+x^2} \\ &= \quad \pi \left[\arctan x \right]_{x:\to 0}^{x:\to\infty} \\ &= \quad \pi \left[\frac{\pi}{2} - 0 \right] \quad = \quad \frac{\pi^2}{2}. \end{aligned}$$

Answer: (D) $\square$

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17. How many real roots does the polynomial $2x^5 + 8x - 7$ have?

(A) None
(B) One
(C) Two
(D) Three
(E) Five

Solution: We have

- $\lim\limits_{x\to-\infty} (2x^5 + 8x - 7) = \lim\limits_{x\to-\infty} 2x^5 = -\infty$, and
- $\lim\limits_{x\to\infty} (2x^5 + 8x - 7) = \lim\limits_{x\to\infty} 2x^5 = \infty$,

so, by the Intermediate Value Theorem, the polynomial $2x^5 + 8x - 7$ has at least one root.

On $x \in \mathbb{R}$, we have:

$$\frac{d}{dx}[2x^5 + 8x - 7] = 10x^4 + 8 > 0,$$

so, by either Rolle's Theorem or the Increasing Test, we conclude that the polynomial $2x^5 + 8x - 7$ has at most one root.

Since the polynomial $2x^5 + 8x - 7$ has at least one *and* at most one root, we see that it has *exactly* one root. Answer: (B) $\square$

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18. Let V be the real vector space of all real 2×3 matrices. Let W be the real vector space of all real 4×1 column vectors. If T is a linear transformation from V <u>onto</u> W, what is the dimension of the subspace $\{\mathbf{v} \in V \,|\, T(\mathbf{v}) = \mathbf{0}\}$ of V?

(A) 2
(B) 3
(C) 4
(D) 5
(E) 6

Solution: By definition of kernel, we have

$$\ker[T] \quad = \quad \{\mathbf{v} \in V \,|\, T(\mathbf{v}) = \mathbf{0}\}.$$

We therefore wish to calculate $\dim(\ker[T])$. We have

- $\dim(\mathrm{dom}\,[T]) = \dim V = 6$ and
- $\dim(\mathrm{im}\,[T]) = \dim W = 4$.

So, since

$$\dim(\ker[T]) + \dim(\mathrm{im}\,[T]) = \dim(\mathrm{dom}\,[T]),$$

we have $\dim(\ker[T]) + 4 = 6$, and so $\dim(\ker[T]) = 2$. Answer: (A) $\square$

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19. Let f and g be twice-differentiable real-valued functions defined on $\mathbb{R}$. Assume, for all $x > 0$, that $f'(x) > g'(x)$. Then which of the following inequalities must be true for all $x > 0$?

(A) $f(x) > g(x)$
(B) $f''(x) > g''(x)$
(C) $[f(x)] - [f(0)] > [g(x)] - [g(0)]$
(D) $[f'(x)] - [f'(0)] > [g'(x)] - [g'(0)]$
(E) $[f''(x)] - [f''(0)] > [g''(x)] - [g''(0)]$

Solution: The example $f(x) = 2x$, $g(x) = x + 1$ shows that (A), (B), (D) and (E) can all fail. By assumption, on $t > 0$, we have $f'(t) > g'(t)$. So, for all $x > 0$, by integrating $f'(t) > g'(t)$, with respect to t, from $t = 0$ to $t = x$, we find that $\int_0^x [f'(t)]\,dt > \int_0^x [g'(t)]\,dt$, and, by the Fundamental Theorem of Calculus, this yields

$$[f(t)]_{t:\to 0}^{t:\to x} \quad > \quad [g(t)]_{t:\to 0}^{t:\to x},$$

or

$$[f(x)] \ - \ [f(0)] \quad > \quad [g(x)] \ - \ [g(0)],$$

which is (C). Answer: (C) $\square$

20. Let f be the function defined on the real line by

$$f(x) = \begin{cases} x/2, & \text{if } x \text{ is rational;} \\ x/3, & \text{if } x \text{ is irrational.} \end{cases}$$

If D is the set of points of discontinuity of f, then D is the

(A) empty set
(B) set of rational numbers
(C) set of irrational numbers
(D) set of nonzero real numbers
(E) set of real numbers

Solution: Because the rationals and irrationals are both dense in $\mathbb{R}$, we see, for all $c > 0$, that $\liminf_{x\to c}[f(x)] = c/3$ and that $\limsup_{x\to c}[f(x)] = c/2$, and, therefore, that f is not continuous at c. Because the rationals and irrationals are both dense in $\mathbb{R}$, we see, for all $c < 0$, that $\liminf_{x\to c}[f(x)] = c/2$ and that $\limsup_{x\to c}[f(x)] = c/3$, and, therefore, that f is not continuous at c. Finally, $\liminf_{x\to 0}[f(x)] \geq 0$ and $\limsup_{x\to 0}[f(x)] \leq 0$ and $f(0) = 0$, and so f is continuous at 0. Answer: (D) □

21. Let P_1 be the set of all primes, $\{2, 3, 5, 7, \ldots\}$, and, for each integer n, let P_n be the set of all prime multiples of n, $\{2n, 3n, 5n, 7n, \ldots\}$. What of the following intersections is nonempty?

(A) $P_1 \cap P_{23}$
(B) $P_7 \cap P_{21}$
(C) $P_{12} \cap P_{20}$
(D) $P_{20} \cap P_{24}$
(E) $P_5 \cap P_{25}$

Solution: For all $p, q \in P_1$, we have $p \neq 23q$. Thus $P_1 \cap P_{23} = \emptyset$.

For all $p, q \in P_1$, we have $p \neq 3q$, and so $7p \neq 21q$. Thus $P_7 \cap P_{21} = \emptyset$.

There exist $p, q \in P_1$ such that $3p = 5q$. (In fact, we can use $p = 5$ and $q = 3$.) Thus there exist $p, q \in P_1$ such that $12p = 20q$. (In fact, we can use $p = 5$ and $q = 3$.) Thus $P_{12} \cap P_{20} \neq \emptyset$. (In fact, $60 \in P_{12} \cap P_{20}$.)

For all $p \in P_1$, we have

$$\text{either} \qquad 2 \nmid p \qquad \text{or} \qquad 3 \nmid p,$$

so

$$\text{either} \qquad 2 \nmid (5p) \qquad \text{or} \qquad 3 \nmid (5p).$$

On the other hand, for all $q \in P_1$, we have

$$\text{both} \qquad 2 \mid (6q) \qquad \text{or} \qquad 3 \mid (6q).$$

So, for all $p, q \in P_1$, we have $5p \neq 6q$, so $20p \neq 24q$. Thus $P_{20} \cap P_{24} = \emptyset$.

For all $p, q \in P_1$, we have $p \neq 5q$, so $5p \neq 25q$. Thus $P_5 \cap P_{25} = \emptyset$.

Answer: (C) □

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22. Let $C(\mathbb{R})$ be the collection of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Then $C(\mathbb{R})$ is a real vector space with vector addition defined by

$$\forall f, g \in C(\mathbb{R}), \forall x \in \mathbb{R}, \qquad (f+g)(x) \quad = \quad [f(x)] \ + \ [g(x)],$$

and with scalar multiplication defined by

$$\forall f \in C(\mathbb{R}), \forall r, x \in \mathbb{R}, \qquad (rf)(x) \quad = \quad r \cdot [f(x)].$$

Which of the following are subspaces of $\mathbb{R}$?

I. $\{f : f \text{ is twice differentiable and } f'' - 2f' + 3f = 0\}$
II. $\{g : g \text{ is twice differentiable and } g'' = 3g'\}$
III. $\{h : h \text{ is twice differentiable and } h'' = h + 1\}$

(A) I only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II and III

Solution: Let V denote the set of all $\phi \in C(\mathbb{R})$ such that ϕ is twice differentiable. Then V is a subspace of $C(\mathbb{R})$.

The map $f \mapsto f'' - 2f' + 3f : V \to C(\mathbb{R})$ is linear and its kernel is I. Thus I is a subspace of V, and, therefore, of $C(\mathbb{R})$.

The map $g \mapsto g'' - 3g' : V \to C(\mathbb{R})$ is linear and its kernel is II. Thus II is a subspace of V, and, therefore, of $C(\mathbb{R})$.

Because the constant function -1 is in III, while -2 is not in III, we see that III is not closed under scalar multiplication. Thus III is NOT a subspace of $C(\mathbb{R})$.

Answer: (B) □

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23. For what value of b is the line $y = 10x$ tangent to the curve $y = e^{bx}$ at some point in the xy-plane?

(A) $10/e$
(B) 10
(C) $10e$
(D) e^{10}
(E) e

Solution: Fix $b \in \mathbb{R}$, and assume that the line $y = 10x$ is tangent to the curve $y = e^{bx}$. Let $x \in \mathbb{R}$ be the first coordinate of the point of tangency. Then

$$10x = e^{bx} \qquad \text{and} \qquad 10 = be^{bx}.$$

Dividing the first equation by the second, we get $(10x)/10 = e^{bx}/(be^{bx})$. That is, $x = 1/b$. Then

$$10/b \quad = \quad 10x \quad = \quad e^{bx} \quad = \quad e^{b(1/b)} \quad = \quad e.$$

Then $b = 10/e$. Answer: (A) $\square$

24. Let h be the function defined by $h(x) = \displaystyle\int_0^{x^2} e^{x+t}\,dt$, for all real numbers x. Then $h'(1) =$

(A) $e - 1$
(B) e^2
(C) $e^2 - e$
(D) $2e^2$
(E) $3e^2 - e$

Solution: For all $x \in \mathbb{R}$, we compute

$$\int_0^{x^2} e^{x+t}\,dt \quad = \quad \left[e^{x+t}\right]_{t:\to 0}^{t:\to x^2} \quad = \quad e^{x+x^2} \;-\; e^x.$$

Thus, for all $x \in \mathbb{R}$, we have $h(x) = e^{x+x^2} - e^x$. Differentiating, we see, for all $x \in \mathbb{R}$, that $h'(x) = (e^{x+x^2})(1 + 2x) - e^x$.

Then $h'(1) = (e^2)(3) - e = 3e^2 - e$. Answer: (E) $\square$

Alternate Solution: Claim: For all $x \in \mathbb{R}$, we have $h(x) = \displaystyle\int_x^{x^2+x} e^s\, ds$.

Proof of claim: Given $x \in \mathbb{R}$. We wish to prove that $h(x) = \displaystyle\int_x^{x^2+x} e^s\, ds$.

Make the change of variables $s = x + t$ and $ds = dt$, in the definition of $h(x)$. This yields $h(x) = \displaystyle\int_x^{x^2+x} e^s\, ds$. *End of proof of claim.*

Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \displaystyle\int_0^x e^s\, ds$. By the claim, for all $x \in \mathbb{R}$, we have $f(x) = [F(x^2 + x)] - [F(x)]$. Differentiating, for all $x \in \mathbb{R}$, we have $f'(x) = [F'(x^2 + x)][2x + 1] - [F'(x)]$. Evaluating this at $x :\to 1$ yields $f'(1) = [F'(2)][3] - [F'(1)]$. By the Fundamental Theorem of Calculus, for all $x \in \mathbb{R}$, $F'(x) = e^x$. Then $F'(2) = e^2$ and $F'(1) = e$. Then $f'(1) = [F'(2)][3] - [F'(1)] = 3e^2 - e$. Answer: (E) □

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25. Let $\{a_n\}_{n=1}^{\infty}$ be defined recursively by $a_1 = 1$ and

$$\text{for all integers } n \geq 1, \qquad a_{n+1} = \left(\frac{n+2}{n}\right) a_n.$$

Then a_{30} is equal to

(A) (15)(31)
(B) (30)(31)
(C) 31 / 29
(D) 32 / 30
(E) [32!] / [(30!)(2!)]

Solution: We have

$$\begin{aligned}
a_{30} &= \left(\frac{31}{29}\right) a_{29} \\
&= \left(\frac{31}{29}\right)\left(\frac{30}{28}\right) a_{28} \\
&= \left(\frac{31}{29}\right)\left(\frac{30}{28}\right)\left(\frac{29}{27}\right) a_{27} \\
&= \cdots \\
&= \left(\frac{31}{29}\right)\left(\frac{30}{28}\right)\left(\frac{29}{27}\right)\cdots\left(\frac{3}{1}\right) a_1 \\
&= \frac{(31!)/(2!)}{29!} \;=\; \frac{(31)(30)}{2} \;=\; (15)(31).
\end{aligned}$$

Answer: (A) □

---

26. For all real x and y, let $f(x,y) = x^2 - 2xy + y^3$. Which of the following is true?

(A) f has all of its relative extrema on the line $x = y$.
(B) f has all of its relative extrema on the parabola $x = y^2$.
(C) f has a relative minimum at $(0,0)$.
(D) f has an absolute minimum at $(2/3, 2/3)$.
(E) f has an absolute minimum at $(1,1)$.

Solution: If a relative extremum of f occurs at a point $(x,y) \in \mathbb{R}^2$, then, by Fermat's Theorem, (x,y) is a critical point of f; in that case, since f is everywhere differentiable, we get $(\partial_1 f)(x,y) = (\partial_2 f)(x,y) = 0$. We have $(\partial_1 f)(x,y) = 2x - 2y$ and $(\partial_2 f)(x,y) = -2x + 3y^2$. Then f has all of its relative extrema on the line $2x - 2y = 0$, *i.e.*, on the line $x = y$. Thus (A) is true.

We have: $[\ 2x - 2y = -2x + 3y^2 = 0\ ]$ iff $[\ x = y$ and $3y^2 = 2x\ ]$ iff $[\ x = y$ and $3x^2 = 2x\ ]$ iff $[\ x = y$ and $(x = 0$ or $x = 2/3)\ ]$ iff $[\ (x,y) = (0,0)$ or $(x,y) = (2/3, 2/3)\ ]$. Thus the two critical points of f are $(0,0)$ and $(2/3,2/3)$.

We calculate that the Hessian of f, evaluated at (x,y), is

$$(Hf)(x,y) := \begin{bmatrix} (\partial_1\partial_1 f)(x,y) & (\partial_1\partial_2 f)(x,y) \\ (\partial_2\partial_1 f)(x,y) & (\partial_2\partial_2 f)(x,y) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 6y \end{bmatrix}.$$

Then

$$(Hf)(0,0) = \begin{bmatrix} 2 & -2 \\ -2 & 0 \end{bmatrix} \quad \text{and} \quad (Hf)(2/3,2/3) = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}.$$

Because $\operatorname{tr}((Hf)(2/3,2/3)) > 0$ and $\det((Hf)(2/3,2/3)) > 0$, we see that $(Hf)(0,0)$ is positive definite. So, by the Second Derivative Test, f attains a relative minimum at $(2/3,2/3)$. Since $(2/3,2/3)$ is NOT on the line $x = y^2$, we see that (B) is false.

Because $\operatorname{tr}((Hf)(0,0)) > 0$ and $\det((Hf)(0,0)) < 0$, we see that $(Hf)(0,0)$ is indefinite. So, by the Second Derivative Test, f has a saddle point at $(0,0)$, and so does not attain a relative extremum at $(0,0)$. Then (C) is false.

Because $\lim\limits_{y\to-\infty} f(0,y) = \lim\limits_{y\to-\infty} y^3 = -\infty$, we see that f has no absolute minimum. Then (D) and (E) are false.

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27. Consider the two planes $x+3y-2z=7$ and $2x+y-3z=0$ in $\mathbb{R}^3$. Which of the following sets is the intersection of these two planes?

(A) $\emptyset$
(B) $\{(0,3,1)\}$
(C) $\{(x,y,z)\,|\,x=t, y=3t, z=7-2t, t\in\mathbb{R}\}$
(D) $\{(x,y,z)\,|\,x=7t, y=3+t, z=1+5t, t\in\mathbb{R}\}$
(E) $\{(x,y,z)\,|\,x-2y-z=-7\}$

Solution: Let P and Q be, respectively, the planes with equations

$$x+3y-2z=7 \qquad \text{and} \qquad 2x+y-3z=0,$$

or, equivalently,

$$(1,3,-2)\cdot(x,y,z)=7 \qquad \text{and} \qquad (2,1,-3)\cdot(x,y,z)=0.$$

The vectors $(1,3,-2)$ and $(2,1,-3)$ are perpendicular to P and Q, respectively. Because $(1,3,-2)$ is not parallel to $(2,1,-3)$, it follows that P and Q are not parallel. Thus $0<\dim(P\cap Q)<2$, and so $\dim(P\cap Q)=1$. This rules out (A), (B) and (E).

We now check (C). Let L be the line described in (C). Replace x, y and z by t, $3t$ and $7-2t$, respectively, in the equation $x+3y-2z=7$. This yields $(t)+3\cdot(3t)-2\cdot(7-2t)=7$, or $(1+9+4)t+(-14)=7$, or $14t-14=7$. It is NOT true, for all $t\in\mathbb{R}$, that $14t-14=7$, and thus $L\not\subseteq P$. This rules out Answer (C).

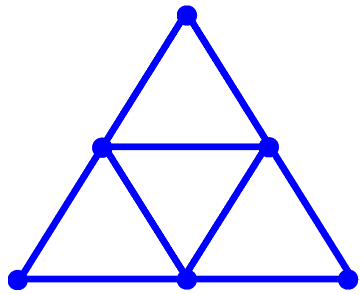
We now check (D). Let M be the line described in D. Replace x, y and z by $7t$, $3+t$ and $1+5t$, respectively, in the two equations

$$x+3y-2z=7 \qquad \text{and} \qquad 2x+y-3z=0.$$

This yields

$$(7t)+3\cdot(3+t)-2\cdot(1+5t)=7 \quad \text{and} \quad 2\cdot(7t)+(3+t)-3\cdot(1+5t)=0,$$

or $(7+3-10)t+(9-2)=7$ and $(14+1-15)t+(3-3)=0$, or $0t+7=7$ and $0t=0$. It IS true, for all $t\in\mathbb{R}$, that $0t+7=7$ and $0t=0$, so $L\subseteq P\cap Q$. So, because $\dim L=1=\dim(P\cap Q)$, we conclude that $L=P\cap Q$. Answer: (D) $\square$



28. The figure above shows an undirected graph with six vertices. Enough edges are to be deleted from the graph in order to leave a spanning tree, which is a connected subgraph having the same six vertices and no cycles. How many edges must be deleted?

(A) One
(B) Two
(C) Three
(D) Four
(E) Five

Solution: The figure separates the plane into five regions – four triangles and the (unbounded) exterior. Eliminating an edge will reduce the separation by one region. A spanning tree will not separate the plane; there will be only one (unbounded) region. Thus we must remove four edges to obtain a spanning tree. Answer: (D) □

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29. For all positive functions f and g of the real variable x, let $\sim$ be a relation defined by

$$f \sim g \qquad \text{if and only if} \qquad \lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = 1.$$

Which of the following is NOT a consequence of $f \sim g$?

(A) $f^2 \sim g^2$
(B) $\sqrt{f} \sim \sqrt{g}$
(C) $e^f \sim e^g$
(D) $f + g \sim 2g$
(E) $g \sim f$

Solution: Since the functions $(\bullet)^2 : \mathbb{R} \to \mathbb{R}$ and $\sqrt{\bullet} : [0,\infty) \to \mathbb{R}$ and $1/\bullet : (0,\infty) \to \mathbb{R}$ are all continuous at 1, we can apply these functions to the equation $\lim\limits_{x\to\infty}\left[\dfrac{f(x)}{g(x)}\right] = 1$, and obtain, respectively,

$$\lim_{x\to\infty}\left[\frac{[f(x)]^2}{[g(x)]^2}\right] = 1, \qquad \lim_{x\to\infty}\left[\frac{\sqrt{f(x)}}{\sqrt{g(x)}}\right] = 1, \qquad \lim_{x\to\infty}\left[\frac{g(x)}{f(x)}\right] = 1,$$

which proves $f^2 \sim g^2$ and $\sqrt{f} \sim \sqrt{g}$ and $g \sim f$, and rules out (A) and (B) and (E).

If we add $\lim\limits_{x\to\infty}\left[\dfrac{f(x)}{g(x)}\right] = 1$ and $\lim\limits_{x\to\infty}\left[\dfrac{g(x)}{g(x)}\right] = 1$, we obtain

$$\lim_{x\to\infty}\left[\frac{[f(x)] + [g(x)]}{g(x)}\right] = 2,$$

and dividing this by 2, we get

$$\lim_{x\to\infty}\left[\frac{[f(x)] + [g(x)]}{2\cdot[g(x)]}\right] = 1,$$

which proves $f + g \sim 2g$, and rules out (D).

Because the function $e^\bullet : \mathbb{R} \to \mathbb{R}$ is continuous at 1, we can apply this function to the equation $\lim\limits_{x\to\infty}\left[\dfrac{f(x)}{g(x)}\right] = 1$, and obtain $\lim\limits_{x\to\infty}\left[\dfrac{e^{f(x)}}{e^{g(x)}}\right] = e$, which proves $e^f \not\sim e^g$. Answer: (C) $\square$

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30. Let f be a function from a set X to a set Y. Consider the following statements.

P : For each $x \in X$, there exists $y \in Y$ such that $f(x) = y$.
Q : For each $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
R : There exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

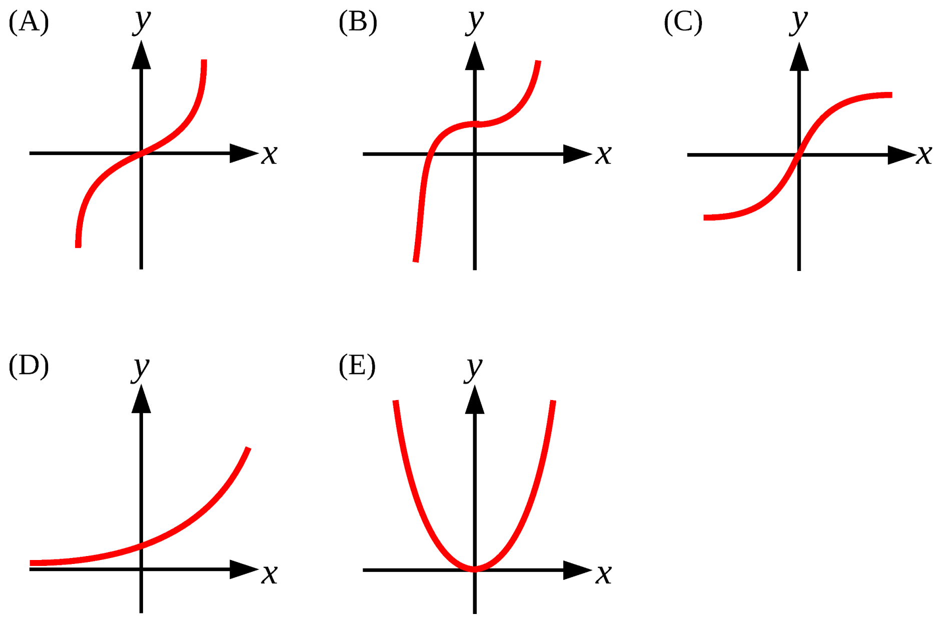
The negation of the statement "f is one-to-one and onto Y" is

(A) P or (not R)
(B) R or (not P)
(C) R or (not Q)
(D) P and (not R)
(E) R and (not Q)

Solution: Statement P is always true.

Statement Q is equivalent to "f is onto Y". Statement R is equivalent to "f is not one-to-one". The negation of "f is one-to-one and onto Y" is "(f is not one-to-one) or (f is not onto Y)." This is equivalent to [R or (not Q)]. Answer: (C) $\square$

31. Which of the following most closely represents the graph of a solution to the differential equation $\frac{dy}{dx} = 1 + y^4$?



Solution: For all $y \in \mathbb{R}$, we have $1 + y^4 \geq 1$. Thus any solution to $\frac{dy}{dx} = 1 + y^4$ will have the property that all of its tangent lines have slope ≥ 1. This rules out (B), (C), (D) and (E). Answer: (A) $\square$

32. Suppose that two binary operations, denoted by $\oplus$ and $\odot$, are defined on a nonempty set S. Suppose, further, that the following conditions are satisfied:

(1) $\forall x, y \in S$, $x \oplus y \in S$ and $x \odot y \in S$.
(2) $\forall x, y, z \in S$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ and $(x \odot y) \odot z = x \odot (y \odot z)$.
(3) $\forall x, y \in S$, $x \oplus y = y \oplus x$.

For each $x \in S$ and each integer $n \geq 1$, the elements $nx, x^n \in S$ are defiend recursively by

- $1x = x^1 = x$ and
- $\forall$integer $k \geq 1$, $(k+1)x = (kx) \oplus x$ and $x^{k+1} = x^k \odot x$.

Which of the following must be true?

I. $\forall x, y \in S$, $\forall$integer $n \geq 1$, $(x \odot y)^n = x^n \odot y^n$.
II. $\forall x, y \in S$, $\forall$integer $n \geq 1$, $n(x \oplus y) = (nx) \oplus (ny)$.
III. $\forall x \in S$, $\forall$integers $m, n \geq 1$, $x^m \odot x^n = x^{m+n}$.

(A) I only
(B) II only
(C) III only
(D) II and III only
(E) I, II and III

Solution: One model of S, $\oplus$, $\odot$ is $\mathbb{R}^{2\times 2}$, matrix addition and matrix multiplication. If we let A and B be any invertible noncommuting matrices in $\mathbb{R}^{2\times 2}$, then $BA \neq AB$, so $A(BA)B \neq A(AB)B$, so $ABAB \neq A^2B^2$. This shows that I is not necessarily true.

Using commutativity and associativity of $\oplus$, for all $x, y \in S$,

$$\begin{aligned} 3(x \oplus y) &= (x \oplus y) \oplus (x \oplus y) \oplus (x \oplus y) \\ &= (x \oplus x \oplus x) \oplus (y \oplus y \oplus y) = (3x) \oplus (3y). \end{aligned}$$

Then II holds for $n = 3$. A similar argument proves II for any specific integer $n \geq 1$. An induction argument proves II for all integers $n \geq 1$.

Using associativity of $\odot$, for all $x \in S$,

$$x^3 \odot x^4 = (x \odot x \odot x) \odot (x \odot x \odot x \odot x) = x^7 = x^{3+4}.$$

Then III holds for $m = 3$ and $n = 4$. A similar argument proves III for any specific integers $m, n \geq 1$. A double induction argument proves III for all integers $m, n \geq 1$. Answer: (D) $\square$

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33. The Euclidean algorithm, described below, is used to find the greatest common divisor (GCD) of two positive integers a and b.

```
INPUT(a)
INPUT(b)
WHILE b > 0
  BEGIN
    r := a mod b
    a := b
    b := r
  END
GCD:=a
OUTPUT(GCD)
```

When the algorithm is used to find the greatest common divisor of $a = 273$ and $b = 110$, which of the following is the sequence of computed values for r?

(A) 2, 26, 1, 0
(B) 2, 53, 1, 0
(C) 53, 2, 1, 0
(D) 53, 4, 1, 0
(E) 53, 5, 1, 0

Solution: Dividing 110 into 273, we obtain a quotient of 2 and a remainder of 53. Using this information, we calculate

$$\begin{aligned} \gcd(273, 110) &= \gcd(110, 273) \\ &= \gcd(110, 273 - 2 \cdot 110) = \gcd(110, 53). \end{aligned}$$

Dividing 53 into 110, we obtain a quotient of 2 and a remainder of 4. Using this information, we calculate

$$\begin{aligned} \gcd(110, 53) &= \gcd(53, 110) \\ &= \gcd(53, 110 - 2 \cdot 53) = \gcd(53, 4). \end{aligned}$$

Dividing 4 into 53, we obtain a quotient of 13 and a remainder of 1. Using this information, we calculate

$$\begin{aligned} \gcd(53, 4) &= \gcd(4, 53) \\ &= \gcd(4, 53 - 13 \cdot 4) = \gcd(4, 1). \end{aligned}$$

Dividing 1 into 4, we obtain a quotient of 4 and a remainder of 1. Using this information, we calculate

$$\begin{aligned}\gcd(4,1) &= \gcd(1,4)\\ &= \gcd(1,4-4\cdot 1) \;=\; \gcd(1,0)\end{aligned}$$

Putting all this together, we get

$$\begin{aligned}\gcd(273,110) &= \gcd(110,53) \;=\; \gcd(53,4)\\ &= \gcd(4,1) \;=\; \gcd(1,0) \;=\; 1.\end{aligned}$$

The remainders calculated were 53, 4, 1 and 0. Answer: (D) □

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34. The minimal distance between any point on the sphere

$$(x-2)^2+(y-1)^2+(z-3)^2=1$$

and any point on the sphere

$$(x+3)^2+(y-2)^2+(z-4)^2=4$$

is

(A) 0
(B) 4
(C) $\sqrt{27}$
(D) $2\left(\sqrt{2}+1\right)$
(E) $3\left(\sqrt{3}-1\right)$

Solution: Call the first sphere S and the second sphere T. Let C be the center of S and let D be the center of T. Let L be the line segment from C to D. Let A be the point of intersection of S and L. Let B be the point of intersection of T and L. The minimal distance from points of S to points of T is $\operatorname{dist}(A,B)$, and this is the number we seek.

Because A and B are points on L, we have

$$\operatorname{dist}(A,B) \;=\; [\operatorname{length}(L)] \;-\; [\operatorname{dist}(C,A)] \;-\; [\operatorname{dist}(D,B)].$$

We have $C=(2,1,3)$ and $D=(-3,2,4)$, so the length of L is

$$\begin{aligned}\operatorname{dist}(C,D) &= \sqrt{(2-(-3))^2+(1-2)^2+(3-4)^2}\\ &= \sqrt{25+1+1} \;=\; \sqrt{27} \;=\; 3\sqrt{3}.\end{aligned}$$

The distance from C to A is the radius of S, which is 1. The distance from D to B is the radius of T, which is 2.

Thus $\operatorname{dist}(A,B)=3\sqrt{3}-1-2=3\left(\sqrt{3}-1\right)$. Answer: (E) □

Alternate Solution: The gradient of $(p-s)^2+(q-t)^2+(r-u)^2$ w.r.t. p,q,r,s,t,u is

$$(\,2(p-s)\,,\,2(q-t)\,,\,2(r-u)\,,\,-2(p-s)\,,\,-2(q-t)\,,\,-2(r-u)\,).$$

The gradients of $(p-2)^2+(q-1)^2+(r-3)^2$ and $(s+3)^2+(t-2)^2+(u-4)^2$ w.r.t. p,q,r,s,t,u are

$$\begin{array}{ll}(\,2(p-2)\,,\,2(q-1)\,,\,2(r-3)\,,\,0\,,\,0\,,\,0\,) & \text{and} \\ (\,0\,,\,0\,,\,0\,,\,2(s+3)\,,\,2(t-2)\,,\,2(u-4)\,) & \end{array}$$

Choose p,q,r,s,t,u so as to minimize

$$(p-s)^2+(q-t)^2+(r-u)^2$$

subject to

$$\begin{array}{rcll}(p-2)^2+(q-1)^2+(r-3)^2 & = & 1 & \text{and} \\ (s+3)^2+(t-2)^2+(u-4)^2 & = & 4 & \end{array}$$

We wish to compute

$$\sqrt{(p-s)^2+(q-t)^2+(r-u)^2}.$$

By Lagrange Multipliers, choose $\lambda,\mu\in\mathbb{R}$ such that

$$(\,2(p-s)\,,\,2(q-t)\,,\,2(r-u)\,,\,-2(p-s)\,,\,-2(q-t)\,,\,-2(r-u)\,)$$

is equal to the λ, μ linear combination of

$$\begin{array}{ll}(\,2(p-2)\,,\,2(q-1)\,,\,2(r-3)\,,\,0\,,\,0\,,\,0\,) & \text{and} \\ (\,0\,,\,0\,,\,0\,,\,2(s+3)\,,\,2(t-2)\,,\,2(u-4)\,). & \end{array}$$

Then

$$\begin{array}{rcll}(\,p-s\,,\,q-t\,,\,r-u\,) & = & \lambda\cdot(\,p-2\,,\,q-1\,,\,r-3\,) & \text{and} \\ (\,s-p\,,\,t-q\,,\,u-r\,) & = & \mu\cdot(\,s+3\,,\,t-2\,,\,u-4\,). & \end{array}$$

Let $A:=(p,q,r)$ and let $B:=(s,t,u)$. Let $C:=(2,1,3)$ and let $D:=(-3,2,4)$. Then $A-B=\lambda(A-C)$ and $B-A=\mu(B-D)$. Let L be the line in $\mathbb{R}^3$ through C and D. Because

$$\begin{array}{rcll}(p-2)^2+(q-1)^2+(r-3)^2 & = & 1 & \text{and} \\ (s+3)^2+(t-2)^2+(u-4)^2 & = & 4, & \end{array}$$

it follows that $\mathrm{dist}(A,C)=1$ and $\mathrm{dist}(B,D)=2$. Then

$$\begin{array}{rcl}\mathrm{dist}(C,D) & = & \sqrt{(2+3)^2+(1-2)^2+(3-4)^2}\ =\ \sqrt{27} \\ & > & 3\ =\ [\mathrm{dist}(A,C)]\ +\ [\mathrm{dist}(B,D)],\end{array}$$

Thus

$$\mathrm{dist}(C,D) \quad > \quad \mathrm{dist}(C,A)+\mathrm{dist}(B,D).$$

On the other hand, by the triangle inequality, we have

$$\mathrm{dist}(C,D) \quad \leq \quad \mathrm{dist}(C,A)+\mathrm{dist}(A,D).$$

Therefore $A \neq B$. Let L be the line through A and B.

Because $A-B=\lambda(A-C)$, it follows that $B=\lambda C+(1-\lambda)A$, and so B is on the line in $\mathbb{R}^3$ through C and A. This line, which passes through A and B, must be L. Then $C \in L$.

Because $B-A=\mu(B-D)$, it follows that $A=\mu D+(1-\mu)B$, and so A is on the line in $\mathbb{R}^3$ through D and B. This line, which passes through A and B, must be L. Then $D \in L$.

Since $C, D \in L$, we conclude that L is the line through C and D. So, since $A \in L$, choose $v \in (0,1)$ such that $A=(1-v)C+vD$. Similarly, since $B \in L$, choose $w \in (0,1)$ such that $B=wC+(1-w)D$.

Then $A-C=v(D-C)$ and $B-D=w(C-D)$. That is,

$$\begin{aligned}(p-2,q-1,r-3) &= v(-5,1,1) \qquad \text{and}\\ (s+3,t-2,u-4) &= w(5,-1,-1).\end{aligned}$$

Recall that

$$\begin{aligned}(p-2)^2+(q-1)^2+(r-3)^2 &= 1 \qquad \text{and}\\ (s+3)^2+(t-2)^2+(u-4)^2 &= 4.\end{aligned}$$

Then

$$\begin{aligned}v^2((-5)^2+1^2+1^2) &= 1 \qquad \text{and}\\ w^2(5^2+(-1)^2+(-1)^2) &= 4.\end{aligned}$$

Then $v=\pm\sqrt{1/27}$ and $w=\pm\sqrt{4/27}$. Let $z:=1/\sqrt{27}$. Then $v=\pm z$ and $w=\pm 2z$. Since $A=(1-v)C+vD$, we see that

$$(p,q,r) \quad = \quad A \quad \in \quad \{\ (1-z)C+zD,\ (1+z)C-zD\ \}.$$

Since $B=wC+(1-w)D$, we see that

$$(s,t,u) \quad = \quad B \quad \in \quad \{\ 2zC+(1-2z)D,\ -2zC+(1+2z)D\ \}.$$

Thus there are two possibilities for (p,q,r) and there are two possibilities for (s,t,u). This gives four possibilities for (p,q,r,s,t,u), each of which satisfies the constraints.

We define $\| \bullet \| : \mathbb{R}^3 \to \mathbb{R}$ by $\|(x, y, z)\| := \sqrt{x^2 + y^2 + z^2}$. We evaluate the objective $(p-s)^2+(q-t)^2+(r-u)^2 = \|A-B\|^2$ assuming

$$\begin{aligned} (p, q, r) &= A = (1-z)C + zD \qquad \text{and} \\ (s, t, u) &= B = 2zC + (1-2z)D \end{aligned}$$

and obtain $\|[(1-z)C + zD] - [2zC + (1-2z)D]\|^2$, which equals

$$\|(1-3z)C + (3z-1)D\|^2 = \|(1-3z)(C-D)\|^2,$$

which equals $(1-3z)^2\|C-D\|^2 = (1-3z)^2\|(5, -1, -1)\|^2 = 27(1-3z)^2$. Evaluating the objective assuming the other three possibilities yields

$$27(1+z)^2 \qquad \text{and} \qquad 27(1-z)^2 \qquad \text{and} \qquad 27(1+3z)^2.$$

Recall: $z = 1/\sqrt{27}$. Among these four possibilities, the smallest value of the objective occurs at the first, so $(p, q, r) = A = (1-z)C + zD$ and $(s, t, u) = B = 2zC + (1-2z)D$. Then $\|A - B\|^2 = 27(1-3z)^2$.

Recall that we wish to compute

$$\sqrt{(p-s)^2 + (q-t)^2 + (r-u)^2},$$

which equals $\|A - B\|$. Thus our final answer is

$$\|A - B\| = \sqrt{\|A - B\|^2} = \sqrt{27(1-3z)^2} = \sqrt{27}(1-3z),$$

which equals

$$(1/z)(1-3z) = (1/z) - 3 = \sqrt{27} - 3 = 3\sqrt{3} - 3 = 3\left(\sqrt{3} - 1\right).$$

Answer: (E) $\square$

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42. Let X and Y be discrete random variables on the set of positive integers. Assume, for each integer $n \geq 1$, that the probability that $X = n$ is 2^{-n}. Assume that Y has the same probability distribution as X, *i.e.*, assume, for each integer $n \geq 1$, that the probability that $Y = n$ is also 2^{-n}. Assume that X and Y are independent. What is the probability that at least one of the variables X and Y is greater than 3?

(A) 1/64
(B) 15/64
(C) 1/4
(D) 3/8
(E) 4/9

Solution: Let $p := \Pr[(X > 3) \text{ or } (Y > 3)]$. We wish to compute p.

Let $q := \Pr[(X \le 3) \,\&\, (Y \le 3)]$. Then $p = 1 - q$. By independence, we have $q = (\Pr[X \le 3])\,(\Pr[Y \le 3])$. We have

$$\Pr[X \le 3] \;=\; \Pr[X \in \{1,2,3\}] \;=\; 2^{-1} + 2^{-2} + 2^{-3} \;=\; 7/8.$$

As X and Y are identically distributed, $\Pr[Y \le 3] = \Pr[X \le 3]$. Then

$$\begin{aligned} q &= (\Pr[X \le 3])\,(\Pr[Y \le 3]) \\ &= (\Pr[X \le 3])^2 \;=\; (7/8)^2 \;=\; 49/64. \end{aligned}$$

Then $p = 1 - q = 1 - (49/64) = 15/64$. Answer: (B) □

Alternate Solution: Let $p := \Pr[(X > 3) \text{ or } (Y > 3)]$. We wish to compute p.

Let $q := \Pr[X > 3]$, $r := \Pr[Y > 3]$, $s := \Pr[(X > 3)\&(Y > 3)]$. By inclusion-exclusion, $p = q + r - s$. Since X and Y are independent, we get $s = rq$. Then $p = q + r - s = q + r - qr$. As X and Y are identically distributed, we get $r = q$. Then $p = q + r - qr = 2q - q^2$.

We have $1 - q = \Pr[X \in \{1,2,3\}] = 2^{-1} + 2^{-2} + 2^{-3} = 7/8$. Then $q = 1/8$. Then $p = 2q - q^2 = (1/4) - (1/64) = 15/64$. Answer: (B) □

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46. Let G be the group of complex numbers $\{1, i, -1, -i\}$ under multiplication. Which of the following statements are true about the homomorphisms of G into itself?

I. $z \mapsto \overline{z}$ defines one such homomorphism, where $\overline{z}$ denotes the complex conjugate of z.
II. $z \mapsto z^2$ defines one such homomorphism
III. For every such homomorphism, there is an integer k such that the homomorphism has the form $z \mapsto z^k$.

(A) None
(B) II only
(C) I and II only
(D) II and III only
(E) I, II and III

Solution: Claim 1: Item I is true. Proof of Claim 1: It is a basic property of complex conjugation that it distributes over complex multiplication. That is, for all $x, y \in \mathbb{C}$, we have $\overline{xy} = \overline{x} \cdot \overline{y}$. Consequently, for all $x, y \in G$, we have $\overline{xy} = \overline{x} \cdot \overline{y}$. That is, $z \mapsto \overline{z} : G \to G$ is a homomorphism of G. *End of proof of Claim 1.*

Claim 2: Item II is true. Proof of Claim 2: Because complex multiplication is commutative, for all $x, y \in \mathbb{C}$, we have

$$(xy)^2 \quad = \quad xyxy \quad = \quad xxyy \quad = \quad x^2 \cdot y^2.$$

Consequently, for all $x, y \in G$, we have $(xy)^2 = x^2 \cdot y^2$. That is, $z \mapsto z^2 : G \to G$ is a homomorphism of G. *End of proof of Claim 2.*

Claim 3: Item I is true. Proof of Claim 3: Let $f : G \to G$ be a homomorphism. We wish to show that there exists $k \in \mathbb{Z}$ such that, for all $z \in G$, we have $f(z) = z^k$.

Because $G = \{i, i^2, i^3, i^4\}$, it follows that, for all $z \in G$, there exists $n \in \{1, 2, 3, 4\}$ such that $z = i^n$. Choose $k \in \{1, 2, 3, 4\}$ such that $f(i) = i^k$. Given $z \in G$. We wish to prove that $f(z) = z^k$.

Choose $n \in \{1, 2, 3, 4\}$ such that $z = i^n$. Then

$$f(z) \quad = \quad f(i^n) \quad = \quad (f(i))^n \quad = \quad (i^k)^n \quad = \quad (i^n)^k \quad = \quad z^k.$$

End of proof of Claim 3.

Answer: (E) □

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49. Up to isomorphism, how many additive Abelian groups G of order 16 have the property that, for all $x \in G$, $x + x + x + x = 0$?

(A) 0
(B) 1
(C) 2
(D) 3
(E) 5

Solution: For any integer $n \geq 1$, let $C_n := \mathbb{Z}/(n\mathbb{Z})$ be the finite additive cyclic group of order n. By the Structure Theorem for Finite Abelian Groups, any additive Abelian group is isomorphic to a direct sum of additive cyclic groups of prime power order. The prime powers that divide 16 are 1, 2, 4 and 16. Consequently, up to isomorphism, the only additive Abelian groups of order 16 are

$$C_{16}, \quad C_8 \oplus C_2, \quad C_4 \oplus C_4, \quad C_4 \oplus C_2 \oplus C_2, \quad C_2 \oplus C_2 \oplus C_2 \oplus C_2.$$

For all $G \in \{C_{16}, C_8 \oplus C_2\}$, there exists $x \in G$ such that $x+x+x+x \neq 0$. For all $G \in \{C_4 \oplus C_4, C_4 \oplus C_2 \oplus C_2, C_2 \oplus C_2 \oplus C_2 \oplus C_2\}$, for all $x \in G$, we have $x + x + x + x = 0$. Thus there are three groups satisfying the conditions of the problem. Answer: (D) □

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59. Let f be an analytic function of a complex variable $z = x + iy$ given by

$$f(z) = (2x + 3y) + i \cdot (g(x, y)),$$

where $g(x, y)$ is a real-valued function of the real variables x and y. If $g(2, 3) = 1$, then $g(7, 3) =$

(A) -14
(B) -9
(C) 0
(D) 11
(E) 18

Solution: Define $Z : \mathbb{R}^2 \to \mathbb{C}$ by $Z(x, y) = x + iy$. Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(x, y) = 2x + 3y$. Then $f \circ Z = h + ig$.

According to the Cauchy-Riemann equations, a counterclockwise $90°$ rotation of $(\partial_1 h, \partial_1 g)$ gives $(\partial_2 h, \partial_2 g)$. That is,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_1 h \\ \partial_1 g \end{bmatrix} = \begin{bmatrix} \partial_2 h \\ \partial_2 g \end{bmatrix}.$$

That is, $-\partial_1 g = \partial_2 h$ and $\partial_1 h = \partial_2 g$.

For all $x, y \in \mathbb{R}$, we have $h(x, y) = 2x + 3y$. Computing partial derivatives, for all $x, y \in \mathbb{R}$, we get $(\partial_1 h)(x, y) = 2$ and $(\partial_2 h)(x, y) = 3$.

So, for all $x, y \in \mathbb{R}$, we have

$$-(\partial_1 g)(x, y) \ = \ 3 \qquad \text{and} \qquad (\partial_2 g)(x, y) \ = \ 2.$$

Multiplying the first equation by -1, and substituting $y :\to 3$, we see, for all $x \in \mathbb{R}$, that $(\partial_1 g)(x, 3) = -3$. So, integrating this equation from $x = 2$ to $x = 7$, we see that

$$[g(7, 3)] \quad - \quad [g(2, 3)] \qquad = \qquad \int_2^7 (-3)\, dx.$$

So, as $\displaystyle\int_2^7 (-3)\, dx = (-3) \cdot (7 - 2) = -15$, we get

$$[g(7, 3)] \quad - \quad [g(2, 3)] \qquad = \qquad -15,$$

which yields $g(7, 3) = [g(2, 3)] - 15$, Then, because $g(2, 3) = 1$, we get $g(7, 3) = 1 - 15 = -14$. Answer: (A) $\square$

NOTE: If we had been asked to compute, say, $g(7,5)$, then we could use a similar technique to calculate

$$[g(7,5)] - [g(7,3)] = \int_3^5 (\partial_2 g)(7,y)\, dy = \int_3^5 2\, dy = 2 \cdot (5-3) = 4.$$

This then gives $g(7,5) = [g(7,3)] + 4 = [-14] + 4 = -10$.

In fact, given the information in the problem, we can compute the value of g at any point in $\mathbb{R}^2$, because we know the partials of g, and, also, the value of g at one point, namely, at $(2,3)$.
