# SOLUTIONS OF VARIATIONS, PRACTICE TEST 2

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$$\begin{aligned} y' + xy(y+2) &= 0 \\ y(0) &= -1 \end{aligned}$$

44-1. Let $y$ be a real-valued function defined on the real line satisfying the initial value problem above. Compute $\lim_{x\to-\infty}[y(x)]$.

*Solution:* Following the notation given in the problem, $y$ and $y(x)$ are used interchangeably. Also, $y'$ and $y'(x)$ are used interchangeably. For all $x \in \mathbb{R}$, $y'(x) = -xy(y+2)$, and so $[y(x) \in \{0,-2\}] \Rightarrow [y'(x) = 0]$. So, by Picard-Lindelöf, exactly one of the following five possibilities holds:

$$\begin{aligned} &\forall x \in \mathbb{R}, y(x) < -2 && \text{or} \\ &\forall x \in \mathbb{R}, y(x) = -2 && \text{or} \\ &\forall x \in \mathbb{R}, -2 < y(x) < 0 && \text{or} \\ &\forall x \in \mathbb{R}, y(x) = 0 && \text{or} \\ &\forall x \in \mathbb{R}, 0 < y(x)\,. \end{aligned}$$

So, as $y(0) = -1$, we get $\forall x \in \mathbb{R}, -2 < y(x) < 0$. Then, for all $x \in \mathbb{R}$,

$$\frac{d}{dx}\left[\frac{y'}{y+2} - \frac{y'}{y}\right] = \frac{-2y'}{y(y+2)} = 2x = \frac{d}{dx}\left[x^2\right].$$

Then

$$\frac{d}{dx}\left[(\ln(y+2)) - (\ln y))\right] = \frac{d}{dx}\left[\frac{y'}{y+2} - \frac{y'}{y}\right] = \frac{d}{dx}\left[x^2\right].$$

Choose $C \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$[\ln(y+2))] - [\ln y] = x^2 + C.$$

Let $K := e^C$. Then, for all $x \in \mathbb{R}$, we have $1+(2/y) = (y+2)/y = Ke^{x^2}$, so $y(x) = y = 2/(Ke^{x^2} - 1)$. Then, because $K = e^C \neq 0$ and because we have $\lim_{x\to-\infty}\left[e^{x^2/2}\right] = \infty$, we conclude that $\lim_{x\to-\infty}[y(x)] = 0$. □

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54-1. Choose a real number $x$ uniformly at random in the interval $[0,3]$. Choose a real number $y$ independently of $x$, and uniformly at random in the interval $[0,4]$. Find the probability that $y < x^2$.

*Solution:* Viewing this as a problem in measure theory, the answer is

$$\frac{\text{area of } \{\ (x,y) \in [0,3] \times [0,4] \mid y < x^2\ \}}{\text{area of } [0,3] \times [0,4]},$$

Let $Q := \{(x,y) \in [0,3] \times [0,4] \,|\, y < x^2\}$. Then the answer is

$$\frac{\text{area of } Q}{12}.$$

Let

$$\begin{aligned} A &:= \{(x,y) \in [0,3] \times [0,4] \,|\, y < x^2 \text{ and } x \leq 2\} \\ B &:= \{(x,y) \in [0,3] \times [0,4] \,|\, y < x^2 \text{ and } x > 2\}. \end{aligned}$$

Then $A \cap B = \emptyset$ and $Q = A \cup B$, so the area of $Q$ is the sum of the areas of $A$ and $B$. We have $A = \{x,y) \in [0,2] \times [0,\infty) \,|\, y < x^2\}$, so the area of $A$ is $\displaystyle\int_0^2 x^2\,dx$. We compute $\displaystyle\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_{x:=0}^{x:=2} = \frac{8}{3}$. Thus the area of $A$ is $8/3$. We have $B = (2,4] \times [0,4]$, so the area of $B$ is 8. Then the area of $Q$ is $(8/3) + 8 = 32/3$.

Then the answer is: $\displaystyle\frac{\text{area of } Q}{12} = \frac{32/3}{12} = \frac{8}{9}$. □

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61-1. A tank initially contains a salt solution of 35 ounces of salt dissolved in 50 gallons of water. Pure water is sprayed into the tank at a rate of 6 gallons per minute. The sprayed water is continually mixed with the salt solution in the tank, and the mixture flows out of the tank at a rate of 2 gallons per minute. If the mixing is instantaneous, how many ounces of salt are in the tank after 12 minutes have elapsed?

*Solution:* For all $t \geq 0$, let $s(t)$ denote the number ounces of salt in the tank at the $t$ minute mark, and let $L(t)$ denote the number of gallons of liquid in the tank at the $t$ minute mark. Then $s(0) = 35$ and $L(0) = 50$. We add 6 gallons per minute and remove 2 gallons per minute, so the net is 4 gallons per minute. Thus, for all $t \geq 0$, we have $L(t) = 50 + 4t$.

We use $s$ and $s(t)$ interchangeably. We also use $s'$ and $s'(t)$ interchangeably. We also use $L$ and $L(t)$ interchangeably. The water sprayed into the tank adds no salt. At any time $t \geq 0$, there are $L$ gallons of liquid in the tank, containing $s$ ounces of salt. So the density of salt in the tank is $s/L$ ounces per gallon. The flow of water out of the tank therefore subtracts $2(s/L)$ ounces of salt per minute. Then, for all $t > 0$, we have $s'(t) = -2s/L = -2s/(50 + 4t) = -s/(25 + 2t)$. Then, for all $t > 0$, we have

$$\frac{d}{dt}[\ln s] \quad = \quad \frac{s'}{s} \quad = \quad \frac{-1}{25+2t} \quad = \quad \frac{d}{dt}\left[-\frac{\ln(25+2t)}{2}\right].$$

Choose $C \in \mathbb{R}$ such that, for all $t \geq 0$, $\ln(s(t)) = [-1/2][\ln(25+2t)]+C$. Let $K := e^C$. Then, for all $t \geq 0$, we have $s(t) = K(25+2t)^{-1/2}$. Then $35 = s(0) = K(25)^{-1/2} = K/5$, so $K = 35 \cdot 5$.

Then $s(12) = 35\cdot 5\cdot (25+2\cdot 12)^{-1/2} = 35\cdot 5\cdot (49)^{-1/2} = 5\cdot 5 = 25$. □

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65-1. Let $g$ be a differentiable function of two real variables, and let $f$ be the function of a complex variable $z$ defined by

$$f(z) = e^{xy} + i \cdot (g(x, y)),$$

where $x$ and $y$ are the real and imaginary parts of $z$, respectively. If $f$ is an analytic function on the complex plane, then $(g(4, 2)) - (g(0, 1)) =$

*Solution:* We will compute $[g(4, 2)] - [g(4, 1)]$ and $[g(4, 1)] - [g(0, 1)]$ separately, and then add the results to get $(g(4, 2)) - (g(0, 1))$.

Define $Z : \mathbb{R}^2 \to \mathbb{C}$ by $Z(x, y) = x + iy$. Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(x, y) = e^{xy}$. Then $f \circ Z = h + ig$.

According to the Cauchy-Riemann equations, a counterclockwise 90° rotation of $(\partial_1 h, \partial_1 g)$ gives $(\partial_2 h, \partial_2 g)$. That is,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_1 h \\ \partial_1 g \end{bmatrix} = \begin{bmatrix} \partial_2 h \\ \partial_2 g \end{bmatrix}.$$

That is, $-\partial_1 g = \partial_2 h$ and $\partial_1 h = \partial_2 g$.

For all $x, y \in \mathbb{R}$, $h(x, y) = e^{xy}$. Computing partial derivatives, for all $x, y \in \mathbb{R}$, we get $(\partial_1 h)(x, y) = ye^{xy}$ and $(\partial_2 h)(x, y) = xe^{xy}$, and so

$$-(\partial_1 g)(x, y) \;=\; xe^{xy} \qquad \text{and} \qquad (\partial_2 g)(x, y) \;=\; ye^{xy}.$$

Multiplying the first equation by $-1$, and substituting $y :\to 1$, we see, for all $x \in \mathbb{R}$, that $(\partial_1 g)(x, 1) = -xe^x$. So, integrating this equation from $x = 0$ to $x = 4$, we get $[g(4, 1)] - [g(0, 1)] = \displaystyle\int_0^4 (-xe^x)\, dx$. We compute the integral on the RHS by integration by parts (differentiating $-x$ w.r.t. $x$, and antidifferentiating $e^x$ w.r.t. $x$), and get

$$\begin{aligned}\int_0^4 (-xe^x)\,dx &= \left[[-xe^x]_{x:\to 0}^{x:\to 4}\right] - \left[\int_0^4 (-e^x)\,dx\right] \\ &= \left[[-xe^x]_{x:\to 0}^{x:\to 4}\right] + \left[\int_0^4 (e^x)\,dx\right] \\ &= [-4e^4 - (-0)] + \left[[e^x]_{x:\to 0}^{x:\to 4}\right] \\ &= [-4e^4] + [e^4 - 1] \;=\; -3e^4 - 1.\end{aligned}$$

Then $[g(4,1)] - [g(0,1)] = -3e^4 - 1$.

Recall that, for all $x, y \in \mathbb{R}$, we have $(\partial_2 g)(x,y) = ye^{xy}$. Substituting $x :\to 4$, we see, for all $y \in \mathbb{R}$, that $(\partial_2 g)(4,y) = ye^{4y}$. So, integrating this equation from $y = 1$ to $y = 2$, we see that

$$[g(4,2)] \quad - \quad [g(4,1)] \qquad = \qquad \int_1^2 (ye^{4y})\,dx.$$

We compute the integral on the RHS by integration by parts (differentiating $y$ w.r.t. $y$, and antidifferentiating $e^{4y}$ w.r.t. $y$), and get

$$\begin{aligned}\int_1^2 (ye^{4y})\,dx &= \left[\left[\frac{ye^{4y}}{4}\right]_{y:\to 1}^{y:\to 2}\right] - \left[\int_1^2 \left(\frac{e^{4y}}{4}\right)\,dy\right] \\ &= \left[\frac{2e^8}{4} - \frac{e^4}{4}\right] - \left[\left[\frac{e^{4y}}{16}\right]_{y:\to 1}^{y:\to 2}\right] \\ &= \left[\frac{2e^8 - e^4}{4}\right] - \left[\frac{e^8}{16} - \frac{e^4}{16}\right] \\ &= \left[\frac{8e^8 - 4e^4}{16}\right] - \left[\frac{e^8 - e^4}{16}\right] \;=\; \frac{7e^8 - 3e^4}{16}.\end{aligned}$$

Then, as $[g(4,2)] - [g(0,1)] = ([g(4,2)] - [g(4,1)]) + ([g(4,1)] - [g(0,1)])$,

$$\begin{aligned}[g(4,2)] - [g(0,1)] &= \left(\frac{7e^8 - 3e^4}{16}\right) + (-3e^4 - 1) \\ &= \frac{7e^8 - 3e^4}{16} + \frac{-48e^4 - 16}{16} \\ &= \frac{7e^8 - 51e^4 - 16}{16}. \quad \square\end{aligned}$$