SOLUTIONS OF VARIATIONS, PRACTICE TEST 3

---

1. If S is a plane in Euclidean 3-space containing $(0,0,0)$, $(2,0,0)$ and $(3,1,1)$, then S is the

(A) xy-plane
(B) xz-plane
(C) yz-plane
(D) plane $y - z = 0$
(E) plane $x + 2y - 2z = 0$

Solution: The xy-plane is $z = 0$ which does not contain $(0,0,1)$, so (A) is not correct. The xz-plane is $y = 0$ which does not contain $(3,1,1)$, so (B) is not correct. The yz-plane is $x = 0$ which does not contain $(2,0,0)$, so (C) is not correct.

The plane $x + 2y - 2z = 0$ is $(1,2,-2) \cdot (x,y,z) = 0$, and we have

$$(1,2,-2) \quad \cdot \quad (3,1,1) \quad = \quad 3 \quad \neq \quad 0,$$

so (E) is not correct.

The plane $y - z = 0$ is $(0,1,-1) \cdot (x,y,z) = 0$, and this contains all three of the points $(0,0,0)$, $(2,0,0)$ and $(3,1,1)$. Answer: (D) □

Date: Printout date: September 19, 2015.

---

2. If a and b are real numbers, which of the following are necessarily true?

I. If $a < b$ and $ab > 0$, then $\frac{1}{a} > \frac{1}{b}$.
II. If $a < b$, then $ac < bc$, for all real numbers $c > 0$.
III. If $a < b$, then $a + c < b + c$, for all real numbers c.
IV. If $a < b$, then $-a > -b$.

Choose one of these answers:

(A) I only
(B) I and III only
(C) III and IV only
(D) II, III and IV only
(E) I,II,III and IV

Solution: If $a < b$ and $ab > 0$, then $\frac{a}{ab} < \frac{b}{ab}$, *i.e.*, $\frac{1}{b} < \frac{1}{a}$, or, equivalently, $\frac{1}{a} > \frac{1}{b}$. Thus I is true.

Also, II, III and IV are all basic facts about the real number system; they are all true. Answer: (E) □

---

3. Compute $\int_0^1 \int_0^y x^3 y^4 \, dx \, dy$.

Solution: We compute

$$\int_0^y x^3 y^4 \, dx = \left[\left(\frac{x^4}{4} \right) y^4 \right]_{x:\to 0}^{x:\to y} = \left[\left(\frac{y^4}{4} \right) y^4 \right] - 0 = \frac{y^8}{4}.$$

Then

$$\begin{aligned} \int_0^1 \int_0^y x^3 y^4 \, dx \, dy &= \int_0^1 \frac{y^8}{4} \, dy \\ &= \left[\frac{y^9}{36} \right]_{x:\to 0}^{x:\to 1} = \left[\frac{1}{36} \right] - 0 = \frac{1}{36}. \end{aligned}$$

□

---

4. For $x \geq 0$, compute $\dfrac{d}{dx}\left(x^\pi \cdot \pi^x\right)$.

Solution: By Logarithmic Differentiation,

$$(d/dx)(\pi^x) = [\pi^x][(d/dx)(x \ln \pi)] = [\pi^x][\ln \pi].$$

Then, using the Product Rule,

$$\begin{aligned} \frac{d}{dx}\left(x^\pi \cdot \pi^x\right) &= \left(\pi x^{\pi-1}\right)\left(\pi^x\right) + \left(x^\pi\right)\left([\pi^x][\ln \pi]\right) \\ &= x^{\pi-1} \cdot \pi^{x+1} + x^\pi \cdot \pi^x \cdot (\ln \pi). \qquad \square \end{aligned}$$

---

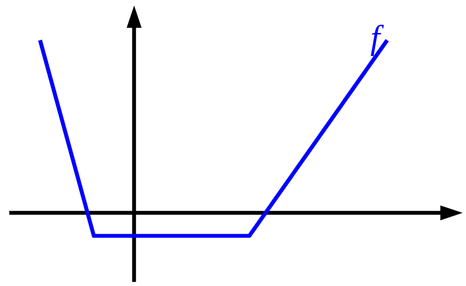
5. Find all functions f defined on the xy-plane such that

$$\frac{\partial}{\partial x}[f(x,y)] = 2x - y \qquad \text{and} \qquad \frac{\partial}{\partial y}[f(x,y)] = x + 2y.$$

Solution: If such a function f were to exist, then we would have

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}[f(x,y)] = \frac{\partial}{\partial x}\frac{\partial}{\partial y}[f(x,y)],$$

yielding $-1 = 1$, a contradiction. Thus no such functions exist. $\square$



6. Sketch the graph of an antiderivative of the function f whose graph is shown in the figure above.

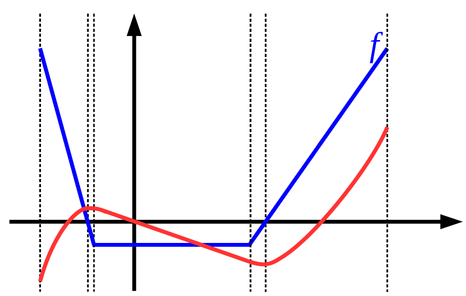
Solution: The graph of f consists, piecewise, of

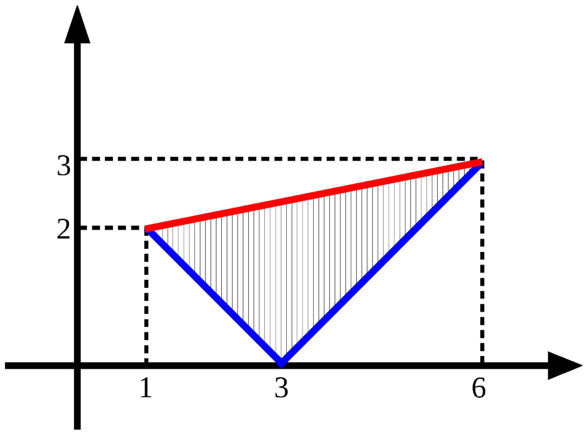
- a very negatively sloped half-line on the left, intersecting the horizontal axis,
- a horizontal line segment in the middle located a bit below the horizontal axis and
- a somewhat positively sloped half-line on the right, intersecting the horizontal axis.

Any antiderivative of f will be, piecewise,

- a very concave down parabolic arc on the left, with a local maximum,
- a line segment in the middle, a bit negatively sloped and
- a somewhat concave up parabolic arc on the right, with a local minimum.

The graph of an antiderivative appears in red in the figure below.





7. Compute the shaded area shown above.

Solution: The line through $(1,2)$ and $(6,3)$ is $y-2=(1/5)(x-1)$, or $y=(1/5)x+(9/5)$. The two line segments

- from $(1,2)$ to $(3,0)$ and
- from $(3,0)$ to $(6,3)$

are both on the graph of $y=|x-3|$. Thus we need to compute the area of the region

between $y=(1/5)x+(9/5)$ and $y=|x-3|$

from $x=1$ to $x=6$. This is $\int_1^6 ((1/5)x+(9/5)-|x-3|)\,dx$, or

$$\left[\int_1^3 \left(\frac{x}{5}+\frac{9}{5}-|x-3|\right)dx\right]+\left[\int_3^6 \left(\frac{x}{5}+\frac{9}{5}-|x-3|\right)dx\right]$$
$$=\left[\int_1^3 \left(\frac{x}{5}+\frac{9}{5}-(3-x)\right)dx\right]+\left[\int_3^6 \left(\frac{x}{5}+\frac{9}{5}-(x-3)\right)dx\right]$$
$$=\left[\int_1^3 \left(\frac{6x}{5}+\frac{-6}{5}\right)dx\right]+\left[\int_3^6 \left(\frac{-4x}{5}+\frac{24}{5}\right)dx\right]$$
$$=\left[\frac{3x^2}{5}+\frac{-6x}{5}\right]_{x:\to 1}^{x:\to 3}+\left[\frac{-2x^2}{5}+\frac{24x}{5}\right]_{x:\to 3}^{x:\to 6}$$
$$=\left[\frac{3\cdot 8}{5}+\frac{-6\cdot 2}{5}\right]+\left[\frac{-2\cdot 27}{5}+\frac{24\cdot 3}{5}\right]=\frac{30}{5}=6. \qquad \square$$

8. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Solution: For all integers $n \geq 1$, we have $1 \leq n^2$, so

$$\frac{n}{n^2+1} \quad \geq \quad \frac{n}{n^2+n^2} \quad = \quad \frac{n}{2n^2} \quad = \quad \frac{1}{2n}.$$

From the integral test for convergence, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$. Multiplying this by $\dfrac{1}{2}$, we get $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} = +\infty$. We conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1} = +\infty$. □