Central Limit Theorem and Finance

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Exercise: $\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n$ 1^∞

Solution: $\left(1+\frac{5}{n}\right)^n \to e^5$, as $n \to \infty$

Exercise: $\lim_{n\to\infty}\left(1+\frac{5}{n}+\frac{10{,}000}{n^2}\right)^n$ 1^∞

Solution: $\left(1+\frac{5}{n}+\frac{10{,}000}{n^2}\right)^n \to e^5$

Exercise: $\lim_{n\to\infty}\left(1+\frac{5}{n}+\frac{10{,}000}{n^2}+\frac{10^{10^{100}}}{n^3}\right)^n$ 1^∞

Solution: $\left(1+\frac{5}{n}+\frac{10{,}000}{n^2}+\frac{10^{10^{100}}}{n^3}\right)^n \to e^5$

Fact: $x_n \sim \frac{5}{n} \quad \Rightarrow \quad (1+x_n)^n \to e^5$

Def'n: Say, $\forall n$, $a_n, b_n > 0$. Then $a_n \sim b_n$ means: $a_n/b_n \to 1$.

$$\frac{1}{2}n^3 + 500n^2 \sim \frac{1}{2}n^3$$

$$\frac{500n^2}{\frac{1}{2}n^3} \to 0$$

$$\frac{5}{n}+\frac{10{,}000}{n^2}+\frac{10^{10^{100}}}{n^3} \sim \frac{5}{n}$$

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Fact: $x_n \sim \frac{5}{n} \quad \Rightarrow \quad (1 + x_n)^n \to e^5$

Exercise: $\lim_{n \to \infty} \left(\frac{n^2}{\left(n + 7\sqrt{n/2}\right)\left(n - 7\sqrt{n/2}\right)} \right)^n$ $\qquad 1^\infty$

Fact: $x_n \sim \frac{5}{n} \quad \Rightarrow \quad (1 + x_n)^n \to e^5$

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Exercise: $$\lim_{n\to\infty} \left(\frac{n^2}{\left(n + 7\sqrt{n/2}\right)\left(n - 7\sqrt{n/2}\right)} \right)^n \qquad 1^\infty$$

Solution: $$\left(\frac{n^2}{\left(n + 7\sqrt{n/2}\right)\left(n - 7\sqrt{n/2}\right)} \right)^n \to e^{7^2/2}$$

Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n^2}{(n + k_n)(n - k_n)} \right)^n \to e^{7^2/2}$

Def'n: Say, $\forall n$, $a_n, b_n > 0$. Then $a_n \sim b_n$ means: $a_n / b_n \to 1$.

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Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n^2}{(n+k_n)(n-k_n)}\right)^n \to e^{7^2/2}$

Exercise: $\lim_{n\to\infty} \left(\frac{n - 7\sqrt{n/2}}{n + 7\sqrt{n/2}}\right)^{7\sqrt{n/2}}$ $\quad 1^\infty$

Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n^2}{(n+k_n)(n-k_n)}\right)^n \to e^{7^2/2}$

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Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n^2}{(n+k_n)(n-k_n)}\right)^n \to e^{7^2/2}$

Exercise: $\lim\limits_{n\to\infty} \left(\frac{n - 7\sqrt{n/2}}{n + 7\sqrt{n/2}}\right)^{7\sqrt{n/2}} \qquad 1^\infty$

Solution: $\left(\frac{n - 7\sqrt{n/2}}{n + 7\sqrt{n/2}}\right)^{7\sqrt{n/2}} \to e^{-7^2}$

Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n-k_n}{n+k_n}\right)^{k_n} \to e^{-7^2}$

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Stirling's Formula: $n! \sim \sqrt{2\pi n}\,(n/e)^n$

Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n-k_n}{n+k_n}\right)^{k_n} \to e^{-7^2}$

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Def'n: Say, $\forall n,\ a_n, b_n > 0$. Then $a_n \sim b_n$ means: $a_n/b_n \to 1$.

$\frac{1}{2}n^3 + 500n^2 \sim \frac{1}{2}n^3$ ← $n :\to n^2 + 2n$ $\qquad n^2 + 2n \to \infty$

$n :\to q_n$

Fact: $x_n \sim y_n,\ q_n \to \infty \quad \Rightarrow \quad x_{q_n} \sim y_{q_n}$

Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n^2}{(n+k_n)(n-k_n)}\right)^n \to e^{7^2/2}$

Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n-k_n}{n+k_n}\right)^{k_n} \to e^{-7^2}$

Stirling's Formula: $n! \sim \sqrt{2\pi n}\,(n/e)^n$ ← $n :\to n^2+2n$

$$(n^2+2n)! \sim \sqrt{2\pi(n^2+2n)}\,((n^2+2n)/e)^{n^2+2n}$$

END OF PRELIMINARIES

Def'n: Say, $\forall n,\ a_n, b_n > 0$. Then $a_n \sim b_n$ means: $a_n/b_n \to 1$.

$\frac{1}{2}n^3 + 500n^2 \sim \frac{1}{2}n^3$ ← $n :\to n^2+2n$ $\qquad n^2+2n \to \infty$

$\frac{1}{2}(n^2+2n)^3 + 500(n^2+2n)^2 \sim \frac{1}{2}(n^2+2n)^3$

Fact: $x_n \sim y_n,\ q_n \to \infty \quad \Rightarrow \quad x_{q_n} \sim y_{q_n}$

Applied Coin-Flipping

$N = 10^{10^{100}}$ N coin flips H heads, T tails

Male height (inches): $69 + 5\frac{H-T}{\sqrt{N}}$

Probability that: $69 - 5 \leq \text{ht} \leq 69 + 5$?

$$69 - 5 \leq 69 + 5\frac{H-T}{\sqrt{N}} \leq 69 + 5$$

$$-5 \leq 5\frac{H-T}{\sqrt{N}} \leq 5$$

DIVIDE BY 5

$$-1 \leq \frac{H-T}{\sqrt{N}} \leq 1$$

Applied Coin-Flipping

$N = 10^{10^{100}}$ $\quad$ N coin flips $\quad$ H heads, T tails

Male height (inches): $69 + 5\frac{H-T}{\sqrt{N}}$

Probability that: $69 - 5 \leq \text{ht} \leq 69 + 5$?

||

Probability that: $-1 \leq \frac{H-T}{\sqrt{N}} \leq 1$?

$$-1 \leq \frac{H-T}{\sqrt{N}} \leq 1$$

Applied Coin-Flipping

$N = 10^{10^{100}}$ N coin flips H heads T tails

Male height (inches): $69 + 5\frac{H-T}{\sqrt{N}}$ square root

Probability that: $69 - 5 \le \text{ht} \le 69 + 5$?

||

Probability that: $-1 \le \frac{H-T}{\sqrt{N}} \le 1$? Answer: $\approx 68\%$

Grav accel (ft/sec^2): $32 + 10^6 \frac{H-T}{N}$ NO square root

< 0.000000000001

Probability that: $32 - \frac{10^6}{\sqrt{N}} \le \text{acc} \le 32 + \frac{10^6}{\sqrt{N}}$?

||

Probability that: $-1 \le \frac{H-T}{\sqrt{N}} \le 1$? Answer: $\approx 68\%$

Applied Coin-Flipping

$N = 10^{10^{100}}$ $\quad$ N coin flips $\quad$ H heads, T tails

---

< 0.000000000001

Probability that: $32 - \frac{10^{8}}{\sqrt{N}} \leq \text{acc} \leq 32 + \frac{10^{8}}{\sqrt{N}}$?

||

Probability that: $-100 \leq \frac{H-T}{\sqrt{N}} \leq 100$?

Answer $> 99.9999999999999\%$

Grav accel (ft/sec^2): $32 + 10^6 \frac{H-T}{N}$ $\quad$ NO square root

< 0.000000000001

Probability that: $32 - \frac{10^{6}}{\sqrt{N}} \leq \text{acc} \leq 32 + \frac{10^{6}}{\sqrt{N}}$?

||

Probability that: $-1 \leq \frac{H-T}{\sqrt{N}} \leq 1$? $\quad$ Answer: $\approx 68\%$

Applied Coin-Flipping

$N = 10^{10^{100}}$ N coin flips H heads, T tails

< 0.000000000001

Probability that: $32 - \frac{10^8}{\sqrt{N}} \leq \text{acc} \leq 32 + \frac{10^8}{\sqrt{N}}$?

||

Probability that: $-100 \leq \frac{H - T}{\sqrt{N}} \leq 100$?

Answer > 99.9999999999999%

Grav accel (ft/sec^2): $32 + 10^6 \frac{H - T}{N}$ NO square root

Maybe all physical phenomena is probabilistic, **but** sometimes the denominator is so large that we can't detect it.

We'll see that $\sqrt{N}$ is the "right" denom. for detectibly probabilistic phenomena.

Coin-Flipping Applied to Finance

$N = 10^{10^{100}}$

Current stock price: 1 USD

30 days from now, derivative contract pays:
5 USD if $1/e <$ price $< e$,   2 USD otherwise
$e \approx 2.71828$

Expected payout?

Grav accel (ft/sec^2): $32 + 10^6 \dfrac{H - T}{N}$ NO square root

---

Maybe all physical phenomena is probabilistic,
but sometimes the denominator is so large
that we can't detect it.

We'll see that $\sqrt{N}$ is the "right" denom.
for detectibly probabilistic phenomena.

Coin-Flipping Applied to Finance

$N = 10^{10^{100}}$ split-second $:= \dfrac{30 \text{ days}}{N}$

Current stock price: 1 USD

ln = 0

30 days from now, derivative contract pays:

5 USD if $1/e <$ price $< e$, 2 USD otherwise

$-1 \leq \ln \leq 1$

Expected payout? Answer: $\approx 5(0.68) + 2(0.32)$

Approx. Black-Scholes

Model: Each split-second, ln(price)

either increases or decreases by $1/\sqrt{N}$,

with 50% chance of uptick, 50% chance of downtick.

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ln(ending price)

Probability that: $-1 \leq \dfrac{H - T}{\sqrt{N}} \leq 1$? Why? Answer: $\approx 68\%$

Coin-Flipping Applied to Finance

Choose an integer $n \geq 1$. *e.g.*: $n = N/2$

ln(stock price) starts at 0 on a (horizontal) number line.

Flip a fair coin $2n$ times in 30 days.

With each head, it
moves $1/\sqrt{2n}$ units in the positive direction (right).

With each tail, it
moves $1/\sqrt{2n}$ units in the negative direction (left).

---

e.g.: $n = 9$
$2n = 18$

Probability that: $-1 \leq \dfrac{H - T}{\sqrt{N}} \leq 1$? Why? Answer: $\approx 68\%$

Coin-Flipping Applied to Finance

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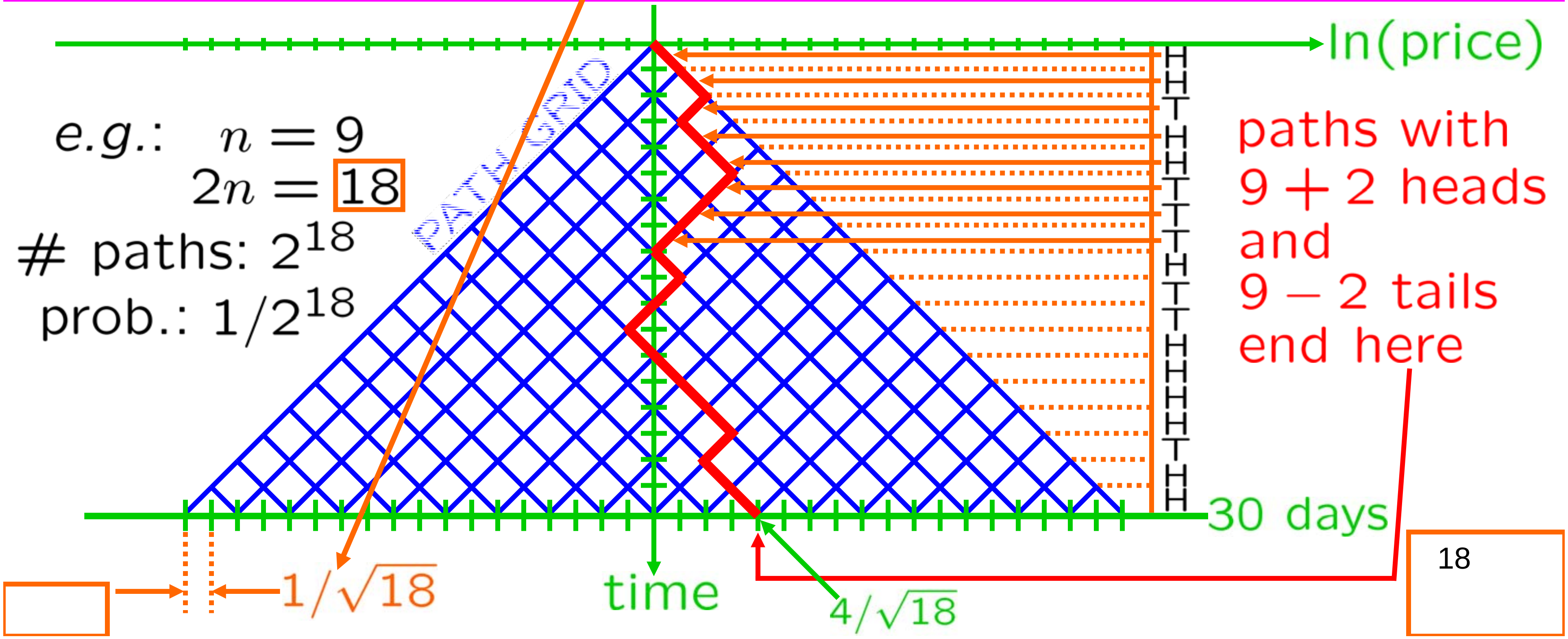
PATH



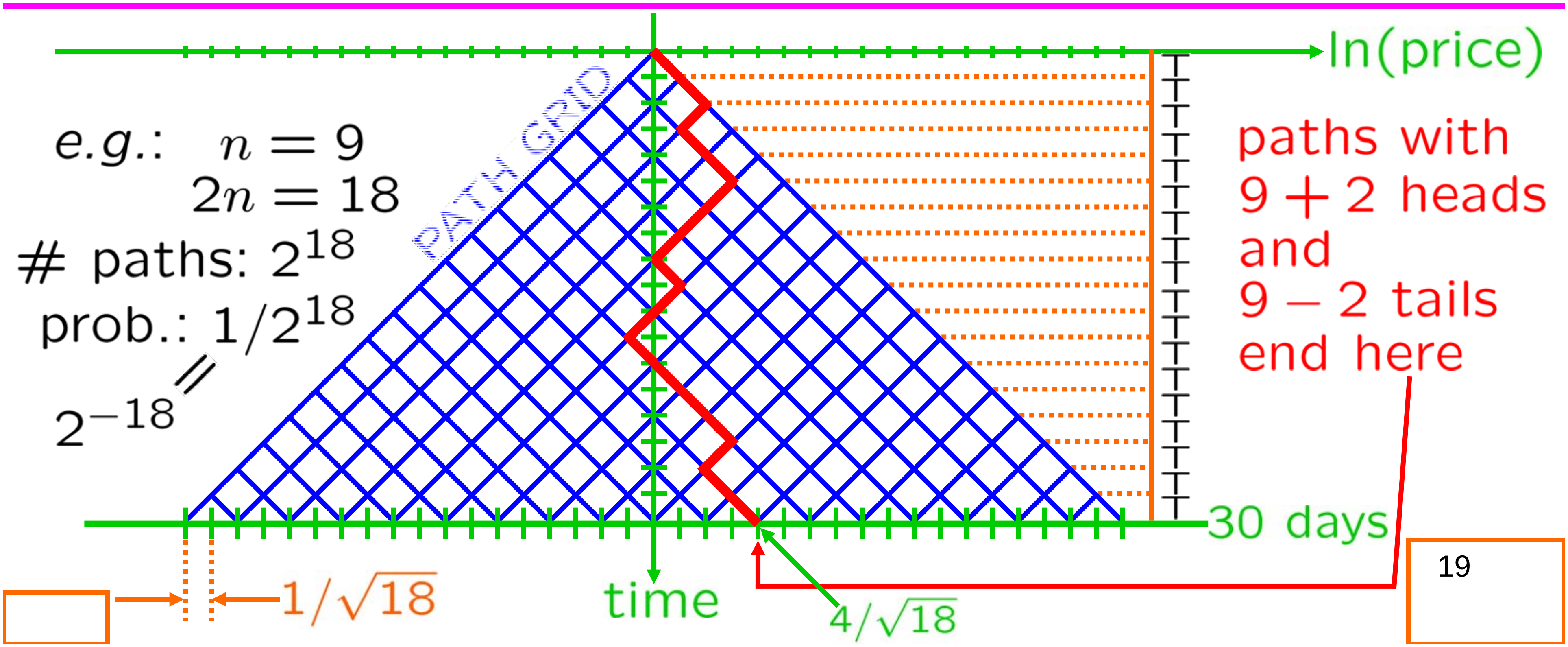
Question: What is the probability of ending at $4/\sqrt{18}$?

Answer: $2^{-18} \times$ # of paths ending at $4/\sqrt{18}$.

$\underbrace{\text{\# of paths ending at } 4/\sqrt{18}}_{\text{\# of collections of 18 Hs and Ts with } 9+2 \text{ Hs and } 9-2 \text{ Ts}}$

$$\underbrace{\text{\# of collections of 18 Hs and Ts with } 9+2 \text{ Hs and } 9-2 \text{ Ts}}_{\binom{18}{9+2}}$$

Start with 18 Ts, choose $9+2$ of them and change the chosen $9+2$ to Hs.



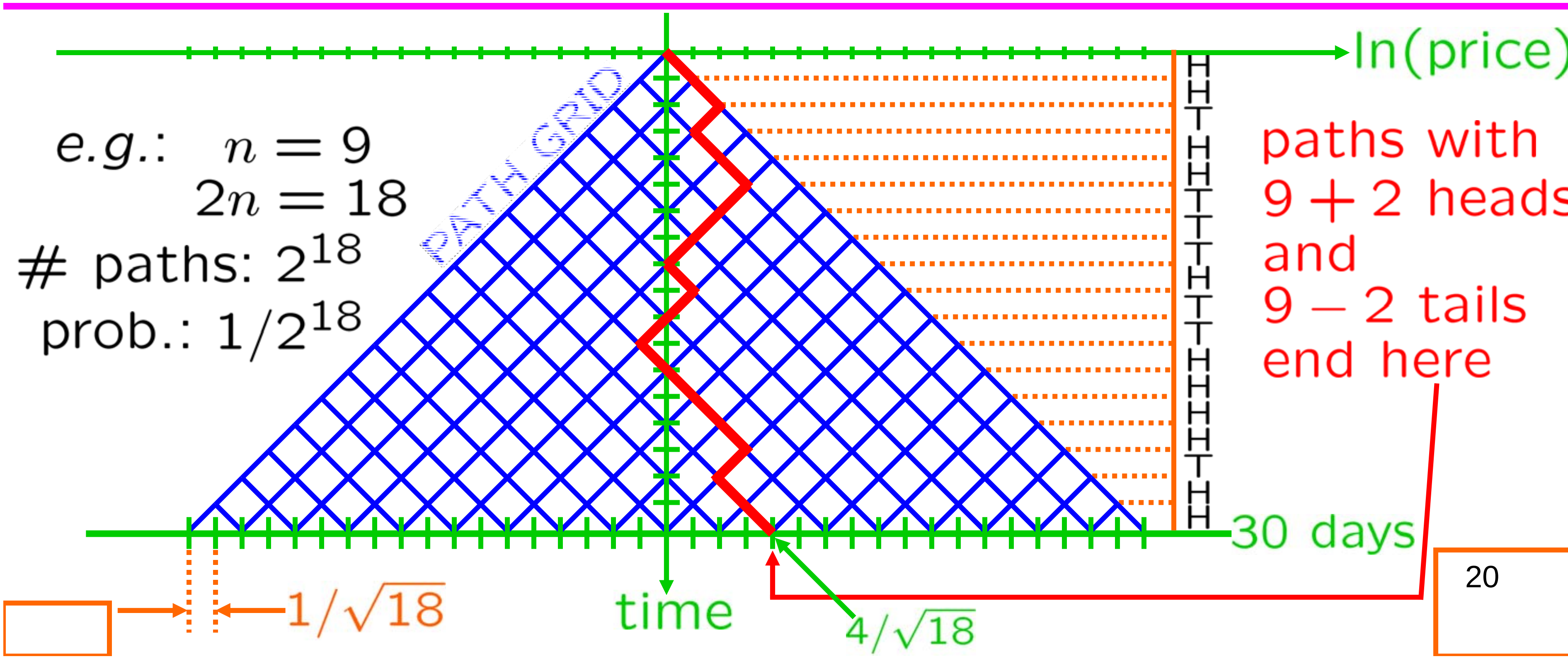
Build a historgram s the probability of ending at $4/\sqrt{18}$?
(a.k.a. “bar graph”)

Answer: 2 ×

$\binom{18}{9+2}$

Question: What is the probabil $\binom{18}{9+2}$ nding at $4/\sqrt{18}$?

Answer: $2^{-18} \times \binom{18}{9+2}$



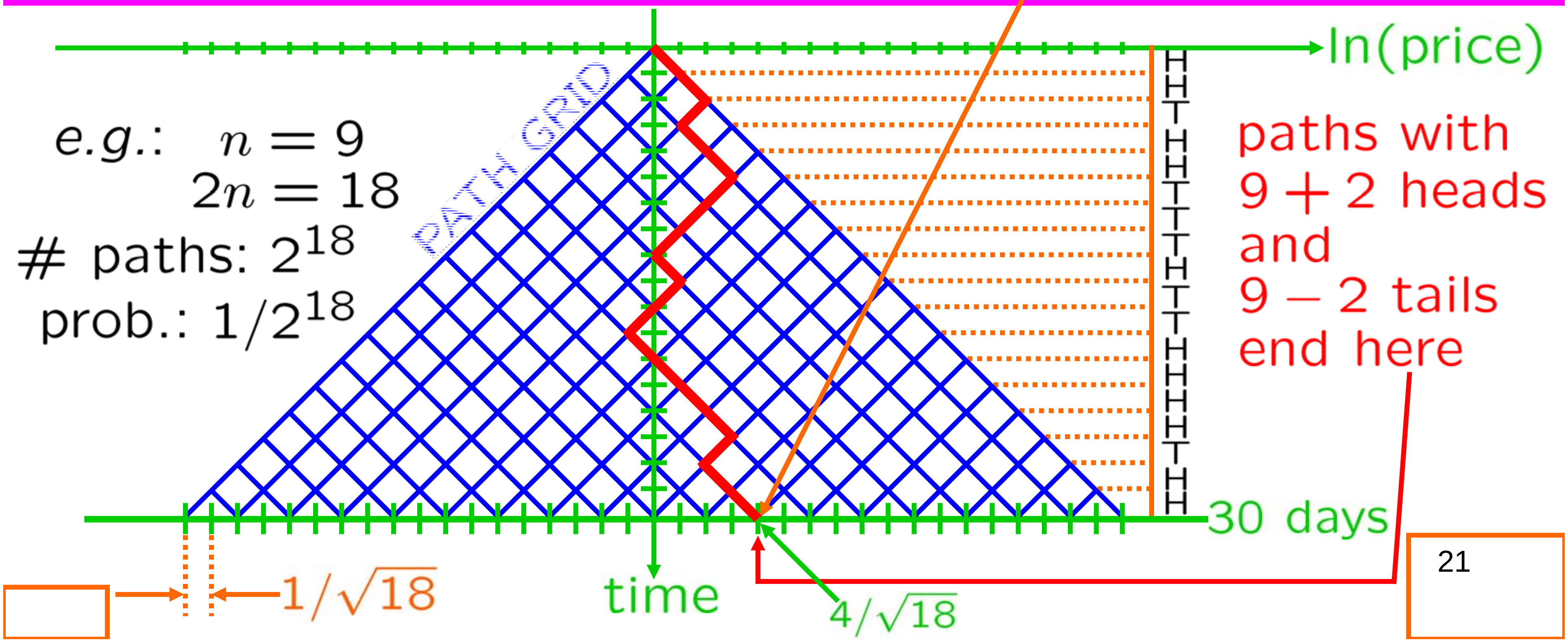
Build a historgram: Start by placing a rectangle ("bar")
(a.k.a. "bar graph") horizontally centered at $4/\sqrt{18}$

height? whose area is $\frac{1}{2^{18}}\binom{18}{9+2}$.

width?
next bar?

Question: What is the probability of ending at $4/\sqrt{18}$?

Answer: $2^{-18} \times \binom{18}{9+2}$



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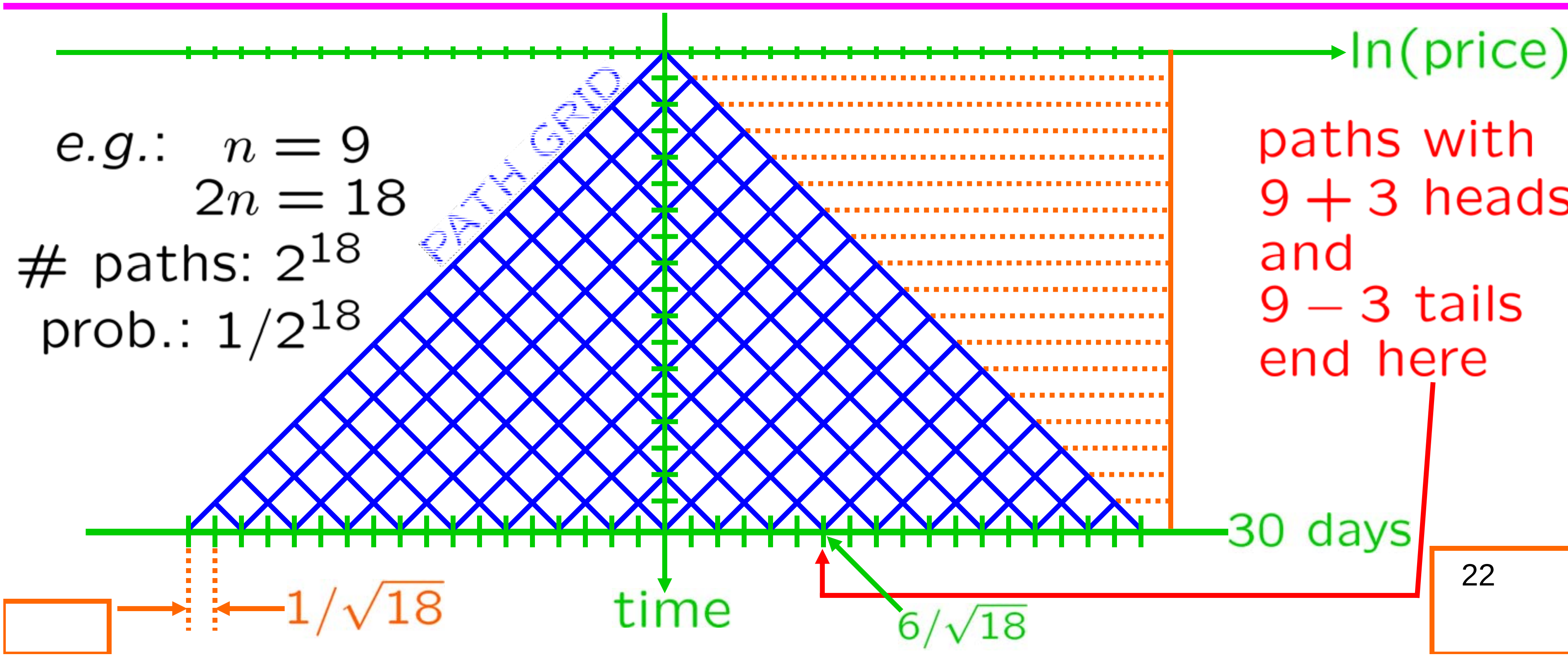


Build a historgram: Start by placing a rectangle ("bar")
(a.k.a. "bar graph")

horizontally centered at $4/\sqrt{18}$

whose area is $\frac{1}{2^{18}}\binom{18}{9+2}$.

height? $\frac{\text{area}}{\text{width}}$

width? $2/\sqrt{18}$

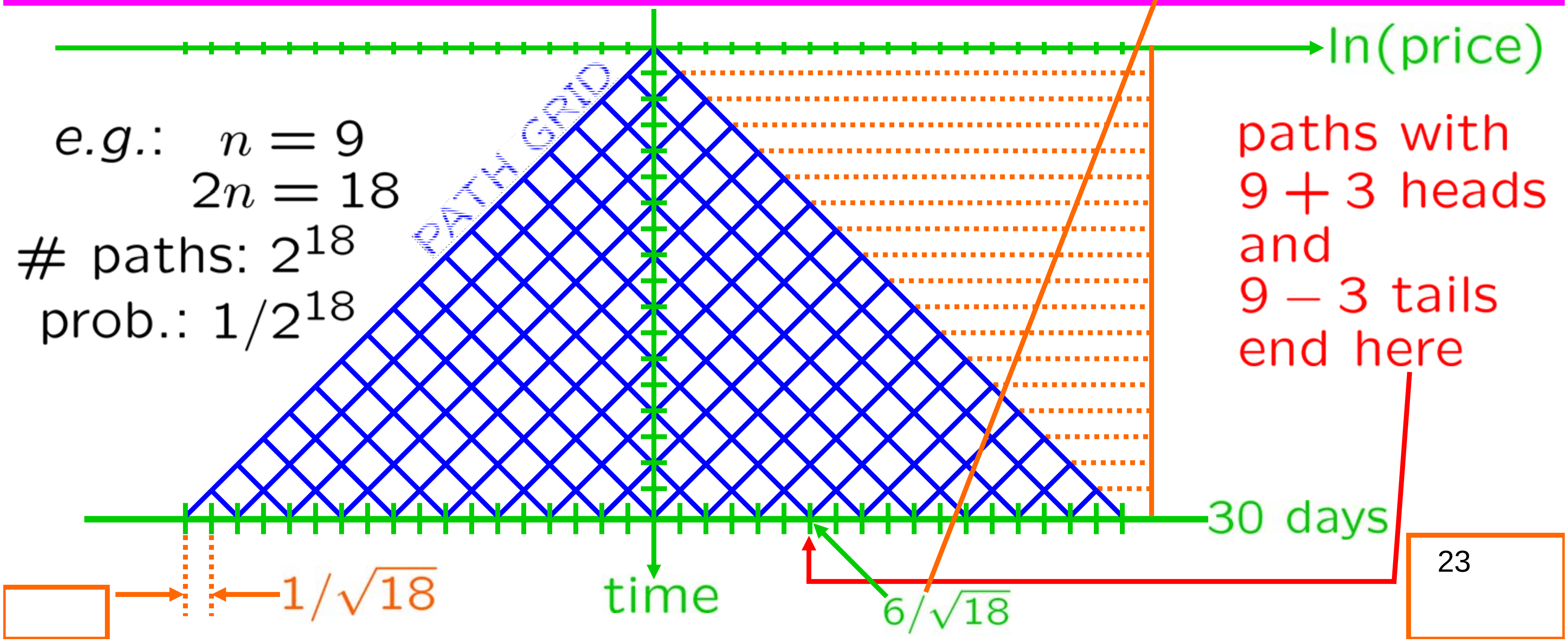
ALL BARS

next bar?

subtract

Next bar: horizontally centered at $6/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+3} = 0.07082$$



Build a historgram: Start by placing a rectangle ("bar") (a.k.a. "bar graph")

horizontally centered at $4/\sqrt{18}$

whose area is $\frac{1}{2^{18}}\binom{18}{9+2} = 0.12140$.

ALL BARS: height? $\frac{\text{area}}{\text{width}}$ width? $2/\sqrt{18}$

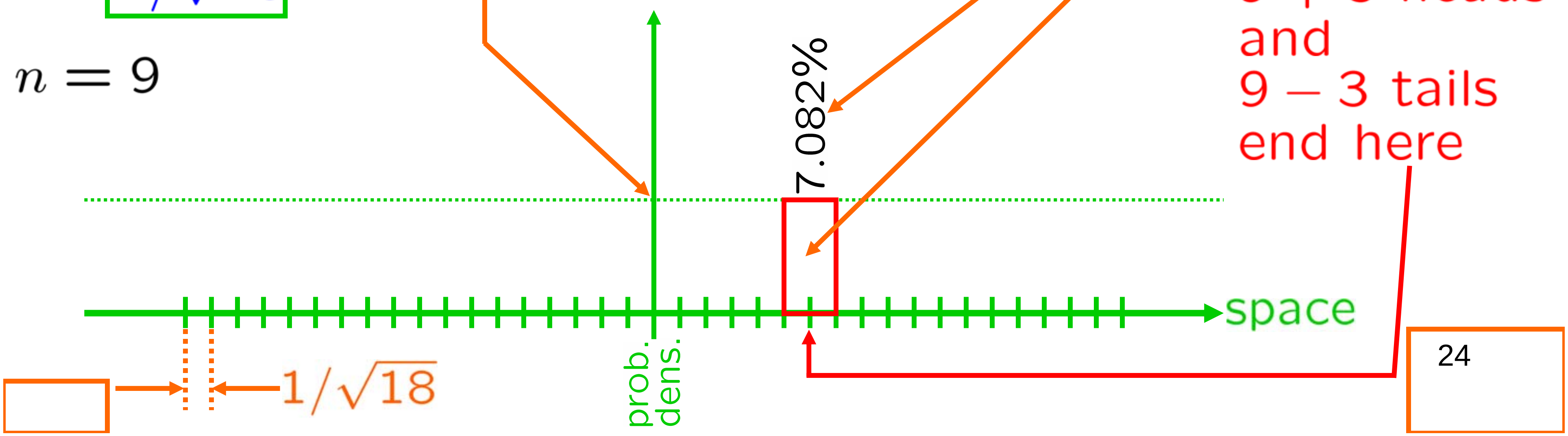
Next bar: horizontally centered at $6/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+3} = 0.07082.$$

Height:

$$\frac{0.07082}{2/\sqrt{18}} = 0.15023$$

$n = 9$



Build a historgram: Start by placing a rectangle (“bar”)
(a.k.a. “bar graph”)

horizontally centered at $4/\sqrt{18}$

height? $\dfrac{\text{area}}{\text{width}}$
width? $2/\sqrt{18}$
ALL BARS

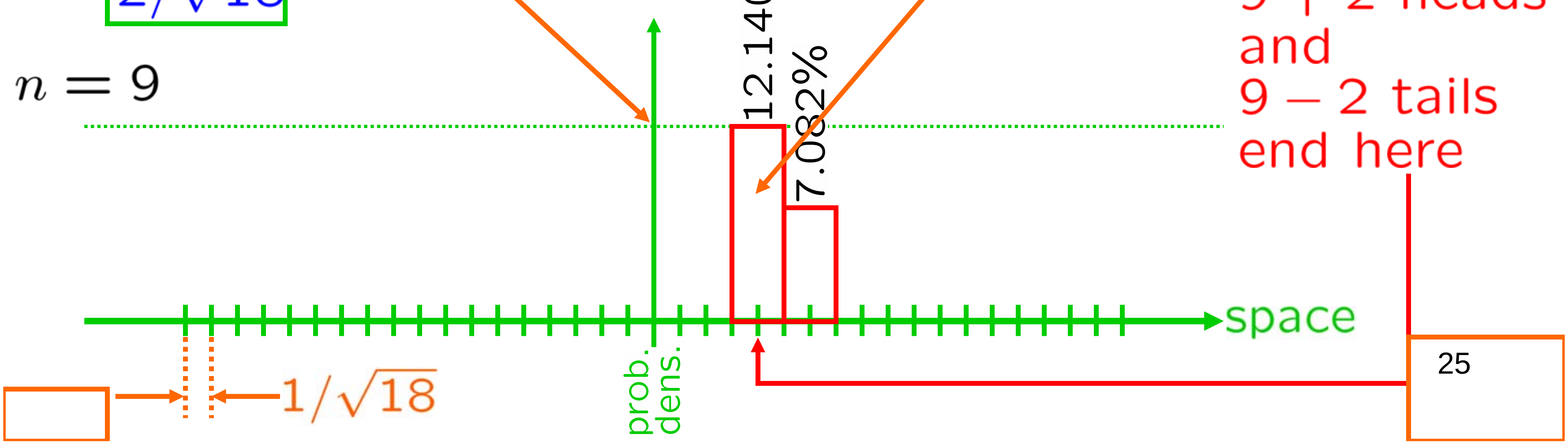
whose area is $\dfrac{1}{2^{18}}\dbinom{18}{9+2} = 0.12140$.

Next bar: horizontally centered at $6/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+3}.$$

Height:

$$\frac{0.12140}{2/\sqrt{18}} = 0.35753$$



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horizontally centered at $4/\sqrt{18}$

height? $\frac{\text{area}}{\text{width}}$

width? $2/\sqrt{18}$

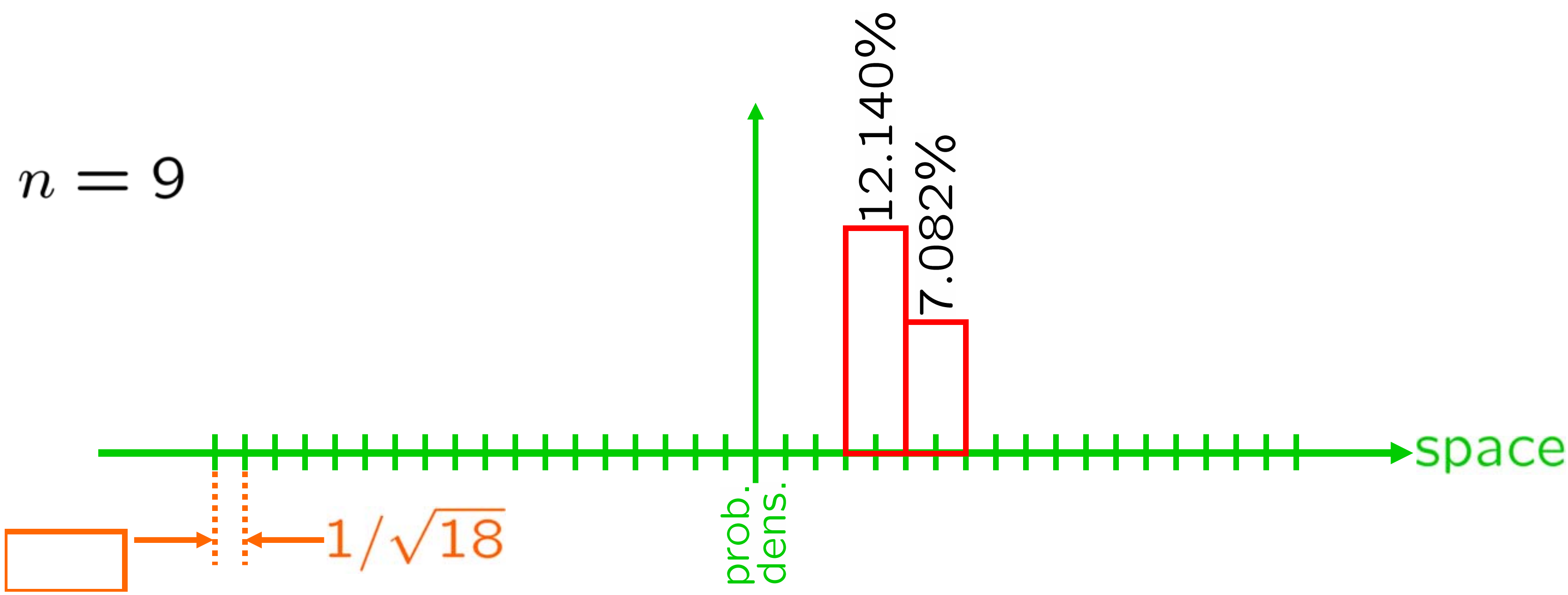
ALL BARS

whose area is $\frac{1}{2^{18}}\binom{18}{9+2}$.

$\forall$integers $k \in [-9, 9]$, make a bar
horizontally centered at $(2k)/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+k}.$$

$n = 9$



Build a historgram (a.k.a. “bar graph”): Start by placing a rectangle (“bar”) horizontally centered at $4/\sqrt{18}$ whose area is $\frac{1}{2^{18}}\binom{18}{9+2}$.

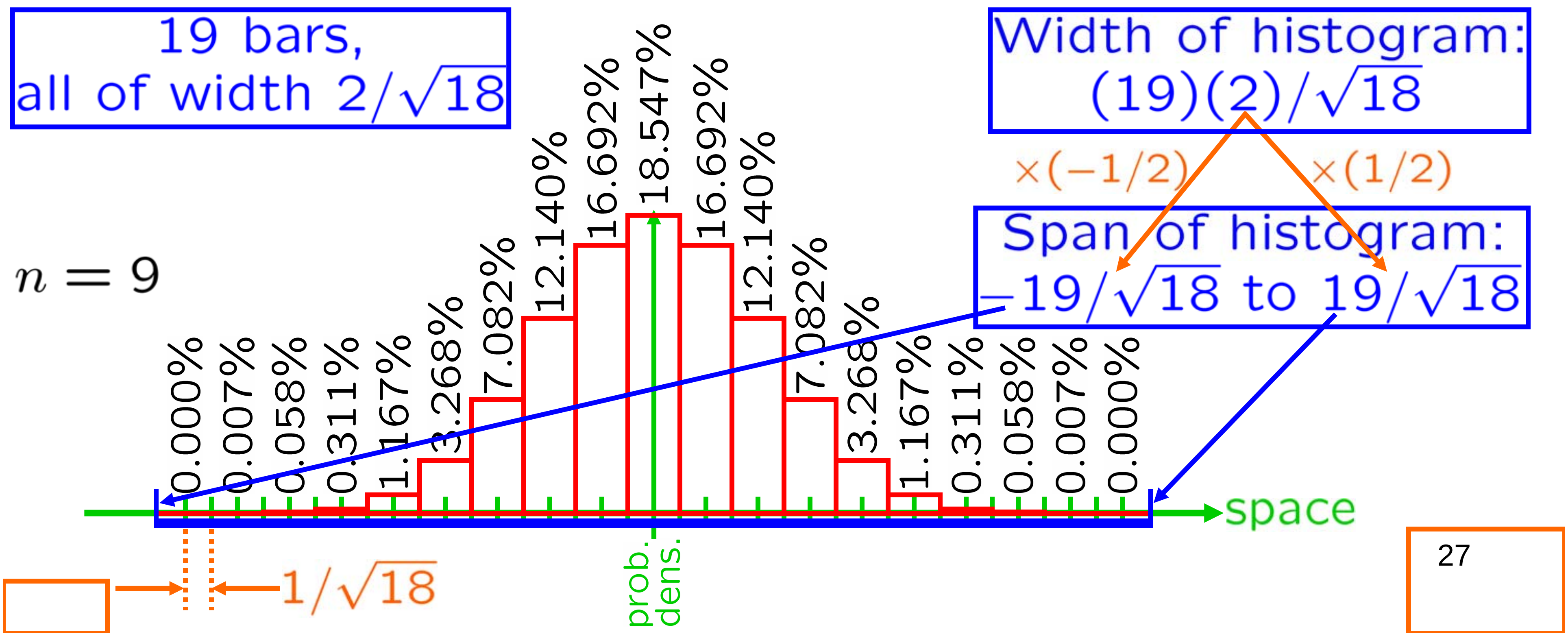
height? $\frac{\text{area}}{\text{width}}$

width? $2/\sqrt{18}$

ALL BARS

$\forall$integers $k \in [-9, 9]$, make a bar horizontally centered at $(2k)/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}}\binom{18}{9+k}.$$



Build a sequence of historgrams: $\forall$integers $n \geq 1$,

$\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

Width of the bars: $2/\sqrt{2n}$

$\forall$integers $k \in [-9, 9]$, make a bar horizontally centered at $(2k)/\sqrt{18}$

$$\text{area} = \frac{1}{2^{18}} \binom{18}{9+k}.$$

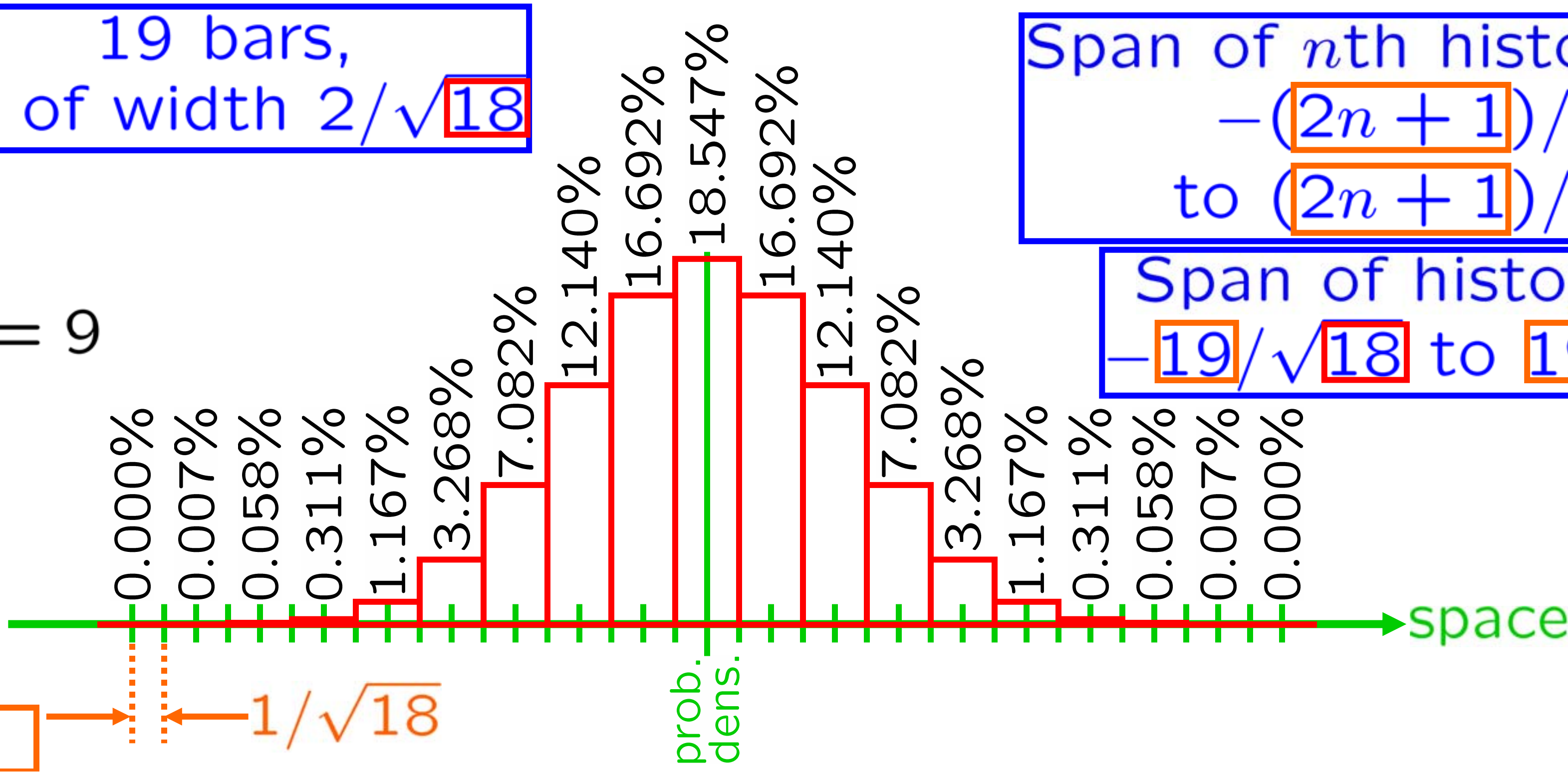
---

19 bars, all of width $2/\sqrt{18}$

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$

Span of histogram: $-19/\sqrt{18}$ to $19/\sqrt{18}$

$n = 9$



$1/\sqrt{18}$

Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}}\binom{2n}{n+k}.$$

Width of the bars: $2/\sqrt{2n}$

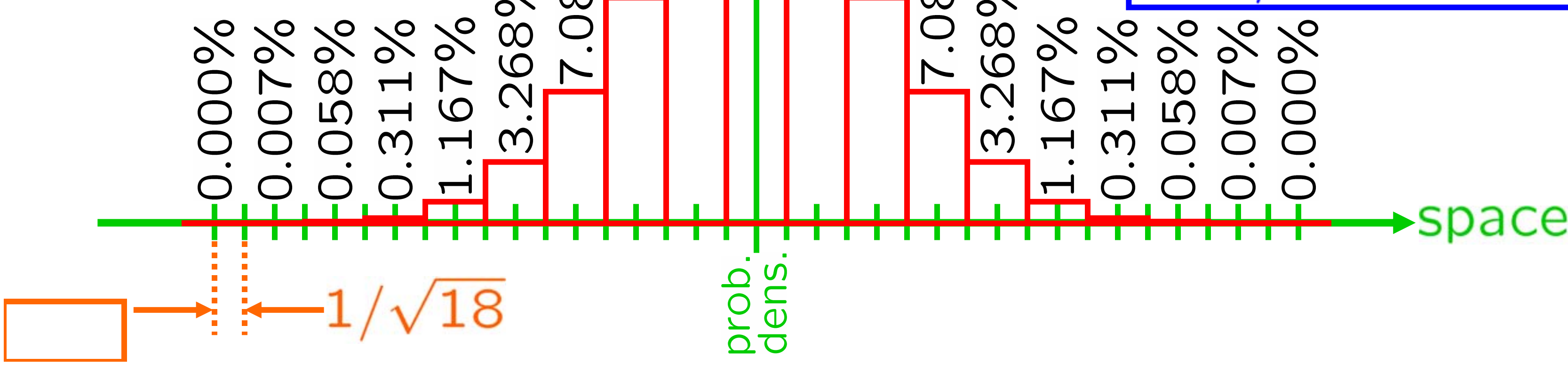
Probability that: $-1 \leq \dfrac{H-T}{\sqrt{N}} \leq 1$? $N = 10^{10^{100}}$

Answer: $\left(\begin{array}{c}\text{Area under hist. } \#N/2 \\ \text{between } x=-1 \text{ and } x=1\end{array}\right) \overset{?}{\approx} 68\%$

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$

Span of histogram: $-19/\sqrt{18}$ to $19/\sqrt{18}$

$n = 9$



Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

Width of the bars: $2/\sqrt{2n}$

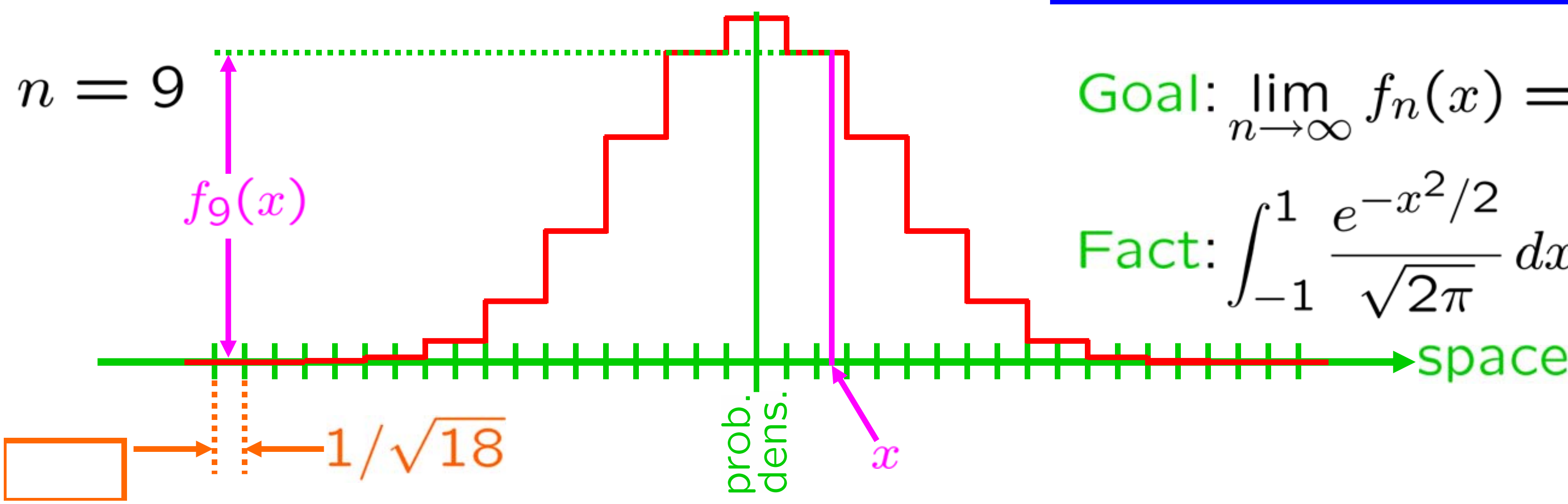
$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

Probability that: $-1 \leq \frac{H-T}{\sqrt{N}} \leq 1$? $N = 10^{10^{100}}$

Answer: $\left(\begin{array}{c}\text{Area under hist. } \#N/2 \\ \text{between } x=-1 \text{ and } x=1\end{array}\right) \overset{?}{\approx} 68\%$

Idea: Find $\lim_{n\to\infty} f_n(x)$, then integrate from $x=-1$ to $x=1$.

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$



Goal: $\lim_{n\to\infty} f_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

Fact: $\int_{-1}^{1} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \approx 0.68$

Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

Width of the bars: $2/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

$n \geq x^2/2$

$\Downarrow$?

x in the span of nth histogram

left side of bar: $[(2k)/\sqrt{2n}] - [1/\sqrt{2n}]$

right side of bar: $[(2k)/\sqrt{2n}] + [1/\sqrt{2n}]$

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$

Goal: $\lim_{n \to \infty} f_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

$n \geq x^2/2$

Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

Width of the bars: $2/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

$n \geq x^2/2$

$\Downarrow$?

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Goal: $\lim_{n \to \infty} f_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

$n \geq x^2/2 \Leftrightarrow -\sqrt{2n} \leq x \leq \sqrt{2n}$

$2n \geq x^2$

$x^2 \leq 2n$ TAKE SQUARE ROOT

Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

Width of the bars: $2/\sqrt{2n}$

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☺ $\Downarrow$?

x in the span of nth histogram

left side of bar: $[(2k)/\sqrt{2n}] - [1/\sqrt{2n}]$

right side of bar: $[(2k)/\sqrt{2n}] + [1/\sqrt{2n}]$

Span of nth histogram: $-(2n+1)/\sqrt{2n}$ to $(2n+1)/\sqrt{2n}$

Goal: $\lim_{n\to\infty} f_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

$n \geq x^2/2 \Leftrightarrow -\sqrt{2n} \leq x \leq \sqrt{2n} \Rightarrow$ x is in the span of the nth histogram

$-\sqrt{2n} = -(2n)/\sqrt{2n} > -(2n+1)/\sqrt{2n}$

$\sqrt{2n} = (2n)/\sqrt{2n} < (2n+1)/\sqrt{2n}$

Build a sequence of historgrams: $\forall$integers $n \geq 1$, $\forall$integers $k \in [-n, n]$, make a bar horizontally centered at $(2k)/\sqrt{2n}$

$$\text{area} = \frac{1}{2^{2n}} \binom{2n}{n+k}.$$

Width of the bars: $2/\sqrt{2n}$

$n \geq x^2/2$
$\Downarrow$
x in the span of nth histogram

$k \to k_n$

left side of bar: $[(2k)/\sqrt{2n}] - [1/\sqrt{2n}]$

right side of bar: $[(2k)/\sqrt{2n}] + [1/\sqrt{2n}]$

$k \to k_n$

height of kth bar in nth histogram:

$$\frac{1}{2^{2n}} \binom{2n}{n+k} / (2/\sqrt{2n}) = \frac{1}{2^{2n}} \binom{2n}{n+k} \frac{\sqrt{2n}}{2}$$

$k \to k_n$

$\frac{\text{area}}{\text{width}}$

Central Limit Theorem: Let $x \in \mathbb{R}$. $\forall n \geq x^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq x \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$.

$h_n = f_n(x)$

Goal: $\lim_{n \to \infty} f_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

Then $h_n \to \frac{e^{-x^2/2}}{\sqrt{2\pi}}$.

Central Limit Theorem: ~~Let $x \in \mathbb{R}$.~~ $\forall n \geq x^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq x \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \dfrac{1}{2^{2n}} \dbinom{2n}{n+k_n} \dfrac{\sqrt{2n}}{2}$.

Then $h_n \to \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$.

Asymptotics of k_n:

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \le 7 \le [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}]$$

$\sqrt{2n}$

Central Limit Theorem at 7: $\forall n \ge 7^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \le 7 \le [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$.

Then $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

Asymptotics of k_n:

$$[(2k_n)/\sqrt{2n}]-[1/\sqrt{2n}] \le 7 \le [(2k_n)/\sqrt{2n}]+[1/\sqrt{2n}]$$

$\sqrt{2n}$

$$2k_n - 1 \le 7\sqrt{2n} \le 2k_n + 1 \qquad 2k_n - 1 \le 2k_n \le 2k_n + 1$$

$$2k_n - 1 \sim 7\sqrt{2n} \sim 2k_n + 1 \qquad 2k_n - 1 \sim 2k_n \sim 2k_n + 1$$

$$2k_n - 1 \sim 2k_n + 1$$

LEADING TERMS

$$2n + 1 \sim 2n - 1 \qquad n :\to k_n$$

$$k_n \to \infty$$

---

at 7

Central Limit Theorem: $\forall n \ge 7^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}]-[1/\sqrt{2n}] \le 7 \le [(2k_n)/\sqrt{2n}]+[1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$.

Then $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

Asymptotics of k_n:

$$2k_n - 1 \sim 7\sqrt{2n} \sim 2k_n + 1$$

$$\begin{array}{ccccc} 2k_n - 1 & \sim & 7\sqrt{2n} & \sim & 2k_n + 1 \\ \| & & & & \| \\ 2k_n - 1 & \sim & 2k_n & \sim & 2k_n + 1 \end{array}$$

Central Limit Theorem at 7: $\forall n \geq 7^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq 7 \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}} \binom{2n}{n + k_n} \frac{\sqrt{2n}}{2}$.

Then $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

Asymptotics of k_n:

$$2k_n - 1 \sim 7\sqrt{2n} \sim 2k_n + 1$$

$$\| \qquad\qquad\qquad\qquad //$$

$$2k_n - 1 \sim 2k_n \sim 2k_n + 1$$

divide by 2: $2k_n \sim 7\sqrt{2n}$

$$k_n \sim 7\sqrt{2n}/2 = 7\sqrt{2n/4} = 7\sqrt{n/2}$$

$\sqrt{4}$

Central Limit Theorem at 7: $\forall n \geq 7^2/2$, choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq 7 \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}} \binom{2n}{n + k_n} \frac{\sqrt{2n}}{2}$.

Then $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$ Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$

$$k_n \sim \qquad 7\sqrt{n/2}$$

Central Limit Theorem at 7: $\forall n \geq 7^2/2,$ choose k_n s.t.

$$[(2k_n)/\sqrt{2n}] - [1/\sqrt{2n}] \leq 7 \leq [(2k_n)/\sqrt{2n}] + [1/\sqrt{2n}],$$

and let $h_n := \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$

Then $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$ ← WANT

$$\underbrace{k_n \sim 7\sqrt{n/2}}_{\text{DIVIDE BY } n}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$\frac{k_n}{n} \sim \frac{7\sqrt{n/2}}{n} \to 0$$

$$x_n \sim y_n \to z \Rightarrow \frac{x_n}{y_n} \to 1, \ y_n \to z \Rightarrow \left[\frac{x_n}{\cancel{y_n}}\right][\cancel{y_n}] \to [\cancel{1}][z] \Rightarrow x_n \to z$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$ Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \sim \frac{7\sqrt{n/2}}{n} \to 0$$

$x_n \sim y_n \to z$

$x_n \sim y_n \to z \Rightarrow x_n \to z$

$\Rightarrow x_n \to z$

$$\frac{k_n}{n} \to 0$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$ Want: $$h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

ASYMPTOTIC $$\frac{n+k_n}{n} = 1 + \frac{k_n}{n} \to 1$$

ASYMPTOTIC $$\frac{n-k_n}{n} = 1 - \frac{k_n}{n} \to 1$$

ADD AND SUBTRACT 1

$$\frac{k_n}{n} \to 0$$

$$n + k_n \sim n$$

$$n - k_n \sim n$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$\frac{k_n}{n} \to 0$$

$$n + k_n \sim n$$

$$n - k_n \sim n$$

$$h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$$

$$= \frac{1}{2^{2n}} \frac{(2n)!}{[(n+k_n)!][((2n)-(n+k_n))!]} \frac{\sqrt{2n}}{2}$$

$\sqrt{4}$

$$= \frac{1}{2^{2n}} \frac{(2n)!}{[(n+k_n)!][(2n-n-k_n)!]} \sqrt{\frac{2n}{4}}$$

$$= \frac{1}{2^{2n}} \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \sqrt{\frac{n}{2}}$$

$$\binom{B}{A} = \frac{B!}{A!(B-A)!}$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$n + k_n \sim n$$

$$n - k_n \sim n$$

$$h_n = \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$$\frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n$$

$$b_n := n - k_n \sim n$$

$$h_n = \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$$= \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[a_n!][b_n!]} \qquad \sqrt{\frac{n}{2}}$$

$$\sqrt{2\pi(2n)}\,(2n/e)^{2n}$$

$$\sqrt{2\pi a_n}\,(a_n/e)^{a_n} \qquad \sqrt{2\pi b_n}\,(b_n/e)^{b_n} \qquad n :\longrightarrow b_n$$

Stirling's Formula: $n! \sim \sqrt{2\pi n}\,(n/e)^n$

$$x_n \sim y_n \to z \quad \Rightarrow \quad x_n \to z$$

$$x_n \sim y_n \to \infty \quad \Rightarrow \quad x_n \to \infty$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$. **Want:** $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n = \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$$= \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[a_n!][b_n!]} \qquad \sqrt{\frac{n}{2}}$$

$$\sim \frac{1}{2^{2n}} \qquad \frac{\sqrt{2\pi(2n)}\,(2n/e)^{2n}}{\left[\sqrt{2\pi a_n}\,(a_n/e)^{a_n}\right]\left[\sqrt{2\pi b_n}\,(b_n/e)^{b_n}\right]} \qquad \sqrt{\frac{n}{2}}$$

Stirling's Formula: $n! \sim \sqrt{2\pi n}\,(n/e)^n$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$

Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n = \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[(n+k_n)!][(n-k_n)!]} \qquad \sqrt{\frac{n}{2}}$$

$$= \frac{1}{2^{2n}} \qquad \frac{(2n)!}{[a_n!][b_n!]} \qquad \sqrt{\frac{n}{2}}$$

$$\sim \frac{1}{2^{2n}} \qquad \frac{\sqrt{2\pi(2n)}(2n/e)^{2n}}{[\sqrt{2\pi a_n}(a_n/e)^{a_n}][\sqrt{2\pi b_n}(b_n/e)^{b_n}]} \sqrt{\frac{n}{2}}$$

$$= \frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]} \qquad \sqrt{\frac{n}{2}} \qquad \frac{1}{2^{2n}} \qquad \frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \quad \to \infty$$

$$b_n := n - k_n \sim n \quad \to \infty$$

$$h_n \sim \frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]} \sqrt{\frac{n}{2}} \quad \frac{1}{2^{2n}} \frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$$\sim$$

$$\frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]} \quad \sqrt{\frac{n}{2}} \quad \frac{1}{2^{2n}} \quad \frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$k_n \sim 7\sqrt{n/2}$, $h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]} \sqrt{\frac{n}{2}} \qquad \frac{1}{2^{2n}} \frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$$2^{2n} n^{2n} e^{-2n}$$

$$a_n^{a_n} e^{-a_n} \qquad b_n^{b_n} e^{-b_n}$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \;\to \infty$$

$$b_n := n - k_n \sim n \;\to \infty$$

$$h_n \sim \frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]}\sqrt{\frac{n}{2}} \quad \frac{1}{2^{2n}} \frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$$= \sqrt{\frac{2\pi(2n)}{[2\pi a_n][2\pi b_n]}\frac{n}{2}} \quad \frac{1}{2^{2n}} \frac{2^{2n} n^{2n} e^{-2n}}{[a_n^{a_n} e^{-a_n}][b_n^{b_n} e^{-b_n}]}$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$

Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \frac{\sqrt{2\pi(2n)}}{[\sqrt{2\pi a_n}][\sqrt{2\pi b_n}]}\sqrt{\frac{n}{2}} \qquad \frac{1}{2^{2n}}\frac{(2n/e)^{2n}}{[(a_n/e)^{a_n}][(b_n/e)^{b_n}]}$$

$$= \sqrt{\frac{2\pi(2n)}{[2\pi a_n][2\pi b_n]}\frac{n}{2}} \qquad \frac{1}{2^{2n}}\frac{2^{2n}n^{2n}e^{-2n}}{[a_n^{a_n}e^{-a_n}][b_n^{b_n}e^{-b_n}]}$$

$$= \sqrt{\frac{n^2}{[a_n][2\pi b_n]}} \qquad \frac{n^{2n}e^{-2n}}{[a_n^{a_n}e^{-a_n}]\,[b_n^{b_n}e^{-b_n}]}$$

$$= \sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \frac{n^{2n}}{a_n^{a_n}b_n^{b_n}} \quad \frac{e^{-2n}}{e^{-a_n}e^{-b_n}}$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \quad \to \infty$$

$$b_n := n - k_n \sim n \quad \to \infty$$

$$h_n \sim \qquad \sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \frac{n^{2n}}{a_n^{a_n} b_n^{b_n}} \qquad \frac{e^{-2n}}{e^{-a_n} e^{-b_n}}$$

$$\sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \frac{n^{2n}}{a_n^{a_n} b_n^{b_n}} \quad \frac{e^{-2n}}{e^{-a_n} e^{-b_n}}$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}} \binom{2n}{n+k_n} \frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \quad \to \infty$$

$$b_n := n - k_n \sim n \quad \to \infty$$

$$h_n \sim \sqrt{\frac{n^2}{2\pi a_n b_n}} \quad \frac{n^{2n}}{a_n^{a_n} b_n^{b_n}} \quad \frac{e^{-2n}}{e^{-a_n} e^{-b_n}}$$

$$\frac{n^{2n}}{a_n^{n} a_n^{k_n} b_n^{n} b_n^{-k_n}} \quad \frac{e^{-2n}}{e^{-(a_n+b_n)}}$$

$$\frac{n^{2n}}{a_n^{n} b_n^{n}} \quad \frac{b_n^{k_n}}{a_n^{k_n}} \quad \frac{e^{-2n}}{e^{-2n}}$$

$$\left(\frac{n^2}{a_n b_n}\right)^{n} \quad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$a_n + b_n = 2n$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \left(\frac{n^2}{a_n b_n}\right)^n \qquad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$\frac{n}{a_n} \to 1$$

$$\left(\frac{n^2}{a_n b_n}\right)^n \qquad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$$ Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \left(\frac{n^2}{a_n b_n}\right)^n \qquad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$\downarrow$$

$$\sqrt{\frac{1}{2\pi}}$$

$$\frac{n}{a_n} \to 1$$

MULTIPLY TOGETHER: $\frac{n^2}{a_n b_n} \to 1$

$$\frac{n}{b_n} \to 1$$

$k_n \sim 7\sqrt{n/2}$, $\quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \sqrt{\frac{n^2}{2\pi a_n b_n}} \quad \left(\frac{n^2}{a_n b_n}\right)^n \quad \left(\frac{b_n}{a_n}\right)^{k_n}$$

$$\downarrow \qquad \| \qquad \|$$

$$\sqrt{\frac{1}{2\pi}} \quad \left(\frac{n^2}{(n+k_n)(n-k_n)}\right)^n \quad \left(\frac{n-k_n}{n+k_n}\right)^{k_n}$$

$$\downarrow \qquad \downarrow$$

$$e^{7^2/2} \qquad e^{-7^2}$$

Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n^2}{(n+k_n)\,(n-k_n)}\right)^n \to e^{7^2/2}$

Fact: $k_n \sim 7\sqrt{n/2} \quad \Rightarrow \quad \left(\frac{n-k_n}{n+k_n}\right)^{k_n} \to e^{-7^2}$

$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}$. Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}$.

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \underbrace{\sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \left(\frac{n^2}{a_n b_n}\right)^n \qquad \left(\frac{b_n}{a_n}\right)^{k_n}}$$

$$\sqrt{\frac{1}{2\pi}} \qquad \sqrt{\frac{1}{2\pi}}\, e^{7^2/2}\, e^{-7^2}$$

$$e^{7^2/2} \qquad e^{-7^2}$$

$$x_n \sim y_n \to z \Rightarrow x_n \to z$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

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$$h_n \to \sqrt{\frac{1}{2\pi}} \; \underbrace{e^{7^2/2}\, e^{-7^2}}_{e^{(7^2/2)-7^2}}$$

$$x_n \sim y_n \to z \Rightarrow x_n \to z$$

$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}.$ Want: $h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$

$$\frac{k_n}{n} \to 0$$

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$$h_n \to \sqrt{\frac{1}{2\pi}}\; e^{7^2/2}\; e^{-7^2}$$

$$e^{(7^2/2)-7^2}$$

$$e^{\overbrace{(7^2/2)-7^2}^{-7^2/2}}$$

$$x_n \sim y_n \to z \Rightarrow x_n \to z$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

$$h_n \sim \underbrace{\sqrt{\frac{n^2}{2\pi a_n b_n}} \qquad \left(\frac{n^2}{a_n b_n}\right)^n \qquad \left(\frac{b_n}{a_n}\right)^{k_n}}$$

$$h_n \to \sqrt{\frac{1}{2\pi}}\; e^{7^2/2}\, e^{-7^2} = \frac{e^{-7^2/2}}{\sqrt{2\pi}}$$

$$e^{\overbrace{(7^2/2)-7^2}^{-7^2/2}}$$

$$x_n \sim y_n \to z \Rightarrow x_n \to z$$

$$k_n \sim 7\sqrt{n/2}, \quad h_n = \frac{1}{2^{2n}}\binom{2n}{n+k_n}\frac{\sqrt{2n}}{2}. \quad \text{Want: } h_n \to \frac{e^{-7^2/2}}{\sqrt{2\pi}}.$$

$$\frac{k_n}{n} \to 0$$

$$a_n := n + k_n \sim n \to \infty$$

$$b_n := n - k_n \sim n \to \infty$$

SUMMARY:

Coin flipping problems are tractable via CLT, and useful in many applied settings, in particular, finance.

Stirling $\Rightarrow$ CLT

QUESTIONS? COMMENTS?