G conn. real Lie group

$\mathrm{Ad}: G \to GL(\mathfrak{g})$

$\mathrm{Ad}\, g_i \to \infty$ in $GL(\mathfrak{g})$

$\Rightarrow g_i \to \infty$ in G

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converse true if

G simple with

finite center

N. Kowalsky

G simple, $\mathrm{rk} \geq 3$

$G \curvearrowright$ Lorentz mfld (conn, isometric)

$\mathrm{Ad}\ g_i \longrightarrow \infty$

$g_i p \longrightarrow q$

$\Longrightarrow G \curvearrowright q$

N. Kowalsky

G simple, fin. cent.

$G \circlearrowright$ Lorentz mfld

nontrivial, nonproper

$\Downarrow$

$\mathfrak{g} \simeq so(n,1)$ or $so(n,2)$

SA: Similar result for simply conn. gps

G simple

$G \curvearrowright$ Riem. mfld

$\mathrm{Ad}\, g_i \longrightarrow \infty$

$g_i p \longrightarrow q$

$\Longrightarrow G \curvearrowright q$

<u>Goal</u>: Pf of this

G simple

$G \curvearrowright$ Hilbert space

$\mathrm{Ad}\, g_i \longrightarrow \infty$

$g_i p \longrightarrow q$

$\Longrightarrow G \curvearrowright q$

Howe-Moore: simple, fin. cent.
no non 0 invariant

$g_i \to \infty \Longrightarrow g_i p \longrightarrow 0,$
$\forall p$

R. Howe & C. Moore

G simple with
finite center
$\Rightarrow$ any ergodic
G-action with
fin. invariant msr.
is mixing

Key ingredient in
superrigidity

Lie Theory

$$\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$$

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$$

$$\exp : \mathfrak{g} \longrightarrow G$$

$$e^X := \exp X$$

[redacted]

$X, Y \in \mathfrak{g}$ conj $\Rightarrow \mathrm{ad}\, X, \mathrm{ad}\, Y$ conj

$$(\mathrm{Ad}\, g) X = Y \implies$$

$$(\mathrm{Ad}\, g)(\mathrm{ad}\, X)(\mathrm{Ad}\, g)^{-1} = \mathrm{ad}\, Y \implies$$

$\mathrm{ad}\, X : \mathfrak{g} \to \mathfrak{g}$ and $\mathrm{ad}\, Y : \mathfrak{g} \to \mathfrak{g}$ have the same eigenvalues

X-orbit $\longrightarrow$ e^{tX}-orbit

$$\operatorname{ad} X : A \mapsto B \mapsto C \mapsto D \mapsto \cdots$$

$$\Downarrow$$

$$(\operatorname{Ad} e^{tX})A = (e^{t \operatorname{ad} X})A =$$

$$A + tB + \frac{t^2}{2!}C + \frac{t^3}{3!}D + \cdots$$

<u>Cor.</u> $(\operatorname{ad} X)A = \lambda A \Longrightarrow$

$$(\operatorname{Ad} e^{tX})A = e^{t\lambda}A$$

Eig.space Generation

$\mathrm{ad}\, H : \mathfrak{g} \to \mathfrak{g}$ real diagonalizable

$\forall \lambda$, $\mathfrak{g}_\lambda := \lambda$-eig.space

$$k := \left\langle \bigcup_{\lambda \neq 0} \mathfrak{g}_\lambda \right\rangle_{LA}$$

$$\Longrightarrow k \trianglelefteq \mathfrak{g}$$

i.e. $[\mathfrak{g}, k] \subseteq k$

Pf: $\forall \lambda \neq 0, \quad \mathfrak{g}_\lambda \subseteq \mathfrak{k}$

$\therefore [\mathfrak{g}_\lambda, \mathfrak{k}] \subseteq [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$

Want: $[\mathfrak{g}_0, \mathfrak{k}] \subseteq \mathfrak{k}$

Want: $\forall \lambda \neq 0,$

$$[\mathfrak{g}_0, \mathfrak{g}_\lambda] \subseteq \mathfrak{g}_\lambda$$

Jacobi ident: $\forall \lambda, \mu$

$$[\mathfrak{g}_\mu, \mathfrak{g}_\lambda] \subseteq \mathfrak{g}_{\mu+\lambda}$$

Set $\lambda \neq 0$, $\mu = 0$. QED

Part of Jacobson-Morozov

$\mathfrak{g}$ simple

$\operatorname{ad} X : \mathfrak{g} \longrightarrow \mathfrak{g}$ nilp.

$$\Downarrow$$

$\exists\, H, Y \in \mathfrak{g} \ni$:

$\operatorname{ad} X : Y \mapsto H \mapsto X \mapsto 0$

& $\operatorname{ad} H : \mathfrak{g} \longrightarrow \mathfrak{g}$ real diag.

G simple $\Longrightarrow$

$G = [G, G] \Longrightarrow$

$\forall g \in G, \quad \det(\mathrm{Ad}\, g) = 1$

$\mathfrak{g}$ simple $\Longrightarrow$

$\ker(\mathrm{ad}) = \{0\} \Longrightarrow$

$$\boxed{\begin{array}{l}\forall H \in \mathfrak{g} \setminus \{0\}, \\ \quad \mathrm{ad}\, H : \mathfrak{g} \to \mathfrak{g} \text{ is non}\,0\end{array}}$$

Transformation Groups

$G \curvearrowright M$ mfld

$\mathfrak{g} \longrightarrow VF(M)$

$$\left[\ \forall X \in \mathfrak{g},\ \forall p \in M \quad X_p := \frac{d}{dt}\bigg|_{t=0} \left(e^{tX} p\right) \in T_p M \ \right]$$

$\forall g \in G, \quad \forall p \in M,$

$$\begin{array}{c} g_* : T_p M \longrightarrow T_{gp} M \\ \rotatebox{90}{$\in$} \\ X_p \longmapsto ? \end{array}$$

$$g_*(X_p) = ((\mathrm{Ad}\, g)X)_{gp}$$

$$X_q = 0$$

$$\Updownarrow$$

$$\forall t,\ e^{tX} q = q$$

$$G \circlearrowright q$$

$$\Updownarrow$$

$\mathfrak{g}$ vanishes at q

$T_i \to \infty$ in $GL(V)$

$\forall i,\ \det(T_i) = 1$

$$\Downarrow$$

$\exists w \in V \ni:$

$T_i w$ unbdd in V

G simple

$G \circlearrowright$ Riem. mfld

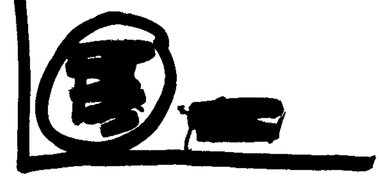
$\mathrm{Ad}\, g_i \longrightarrow \infty$ in $GL(\mathfrak{g})$

$g_i p \longrightarrow q$

✓ <u>Want</u>: $G \circlearrowright q$

Fix $W \in \mathfrak{g}$ $\ni$: $(\mathrm{Ad}\, g_i)W$ unbdd in $\mathfrak{g}$

$$\checkmark \left\{ \begin{array}{l} \forall i, \quad p_i := g_i p \longrightarrow q \\ \qquad W_i := (\mathrm{Ad}\, g_i)W \text{ unbdd} \end{array} \right.$$



Subseq: $W_i \to \infty$ in $\mathfrak{g}$

$\delta_i W_i$ bdd in $\mathfrak{g}$ &
bdd from 0

$\delta_i \longrightarrow 0$ in $\mathbb{R}$

✓ Subseq: $\delta_i W_i \longrightarrow X \neq 0$

$$(g_i)_*(W_p) = \underbrace{(W_i)_{p_i}}_{\text{constant length}}$$

$A_i \longrightarrow 0$ in $\mathbb{R}$

$$A_i\big((W_i)_{p_i}\big) = (A_i W_i)_{p_i}$$

$$\longrightarrow X_q$$

$$\therefore \quad X_q = 0$$

✓ $$\therefore \quad e^{tX} q = q$$

$$(\mathrm{Ad}\, g_i)\, W = W_i$$

$$(\mathrm{Ad}\, g_i)(\mathrm{ad}\, W)(\mathrm{Ad}\, g_i)^{-1} = \mathrm{ad}\, W_i$$

$\mathrm{ad}\, W : \mathfrak{g} \to \mathfrak{g}$ and

$\mathrm{ad}\, W_i : \mathfrak{g} \to \mathfrak{g}$ have

the same eigenvalues

$\operatorname{ad} W_i : \mathfrak{g} \to \mathfrak{g}$

$\{$eig. vals$\}$ const in i

$\lambda_i \to 0$ in $\mathbb{R}$

$$\lambda_i (\operatorname{ad} W_i) = \operatorname{ad}(\lambda_i W_i)$$
$$\to \operatorname{ad} X$$

$\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$

all eig. vals 0

$\therefore$ nilpotent

$\mathrm{ad}\, X : Y \mapsto H \mapsto X \mapsto 0$

✓ $\mathrm{ad}\, H : \mathfrak{g} \to \mathfrak{g}$ real diag.

$X \neq 0 \quad \therefore \quad H \neq 0 \quad \therefore$

✓ $\mathrm{ad}\, H : \mathfrak{g} \to \mathfrak{g}$ non 0

$(\mathrm{Ad}\, e^{tX})\, Y =$

$$Y + tH + \frac{t^2}{2!} X + 0 + 0 + \cdots$$

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[redacted] [redacted]

$$(e^{tX})_*(Y_q) =$$

$$((\mathrm{Ad}\, e^{tX})Y)_{e^{tX}q} =$$

$$\left(Y + tH + \frac{t^2}{2}X\right)_q =$$

$Y_q + tH_q$, $\forall t$ on sphere

$$\therefore\ H_q = 0$$

✓ $\therefore\ e^{tH}q = q$

$\operatorname{ad} H : \mathfrak{g} \longrightarrow \mathfrak{g}$

non0, real diag.

$\forall \lambda$, $\mathfrak{g}_\lambda := \lambda$-eig. sp.

$$k := \left\langle \bigcup_{\lambda \neq 0} \mathfrak{g}_\lambda \right\rangle_{LA} \neq \{0\}$$

$k \trianglelefteq \mathfrak{g} \quad \therefore \quad k = \mathfrak{g}$

<u>Want</u>: k vanishes at q

<u>Want</u>: $\forall \lambda \neq 0$,

$\mathfrak{g}_\lambda$ van. at q

Fix $\lambda \neq 0$,

$S \in \mathfrak{g}_\lambda$

<u>Want</u>: $S_q = 0$

$(\mathrm{ad}\, H)\, S = \lambda S$

$(\mathrm{Ad}\, e^{tH})\, S = e^{t\lambda} S$

$$(e^{tH})_*(S_q) =$$

$$((\mathrm{Ad}\, e^{tH})S)_{e^{tH}q} =$$

$$(e^{t\lambda}S)_q =$$

$$e^{t\lambda} S_q, \quad \forall t \text{ on sphere}$$

$\lambda \neq 0 \quad \therefore \{e^{t\lambda}\} = (0, \infty)$

$S_q = 0$

QED

[redacted]

geometric smooth dynamics

$\downarrow$

topological dynamics on Hilbert space

$\downarrow$

ergodic theory

R. Ellis & M. Nerurkar

~~simple~~

$G \curvearrowright$ Hilb. faithful irreducible

$$\mathrm{Ad}\, g_i \longrightarrow \infty$$

$$\Rightarrow \forall v, w \in \text{Hilb},$$

$$\langle g_i v, w \rangle \longrightarrow 0$$

"decay at Ad - ∞"