Finite Element Exterior Calculus and Applications

Part III

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August 15–18, 2015

Finite element spaces of differential forms

Differential forms on a domain $\Omega \subset \mathbb{R}^n$

- Differential k-forms are functions $\Omega \to \mathrm{Alt}^k\mathbb{R}^n$

  0-forms: functions; 1-forms: covector fields; k-forms: $\binom{n}{k}$ components

  $$u = \sum_\sigma f_\sigma \, dx^\sigma := \sum_{1 \le \sigma_1 < \cdots < \sigma_k \le n} f_{\sigma_1 \cdots \sigma_k} \, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}$$

- The wedge product of a k-form and an l-form is a $(k+l)$-form
- The exterior derivative du of a k-form is a $(k+1)$-form
- A k-form can be integrated over a k-dimensional subset of Ω
- Given $F : \Omega \to \Omega'$, a k-form on Ω' can be pulled back to a k-form on Ω.
- The trace of a k-form on a submanifold is the pull back under inclusion.
- Stokes theorem: $\displaystyle\int_\Omega du = \int_{\partial\Omega} \mathrm{tr}\, u, \quad u \in \Lambda^{k-1}(\Omega)$
- The exterior derivative can be viewed as a closed, densely-defined op $L^2\Lambda^k \to L^2\Lambda^{k+1}$ with domain $H\Lambda^k(\Omega) = \{\, u \in L^2\Lambda^k \mid du \in L^2\Lambda^{k+1} \,\}$. If Ω has a Lipschitz boundary, it has closed range.

The L^2 de Rham complex and its discretization

$$0 \to L^2\Lambda^0 \xrightarrow{d} L^2\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} L^2\Lambda^n \to 0$$

Our goal is to define spaces $V_h^k \subset H\Lambda^k$ satisfying the approximation, subcomplex, and BCP assumptions.

In the case $k = 0$, $V_h^k \subset H^1$ will just be the Lagrange elements. It turns out that for $k > 0$ there are two distinct generalizations.

Finite element spaces

A FE space is constructed by assembling three ingredients: Ciarlet '78

- A *triangulation* $\mathcal{T}$ consisting of polyhedral elements T
- For each T, a space of *shape functions* $V(T)$, typically polynomial
- For each T, a set of *DOFs*: a set of functionals on $V(T)$, each associated to a face of T. These must be *unisolvent*, i.e., form a basis for $V(T)^*$.

The FE space V_h is *defined* as functions piecewise in $V(T)$ with DOFs *single-valued* on faces. The DOFs determine (1) the interelement continuity, and (2) a projection operator into V_h.

The Lagrange finite element space $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ for $H^1 = H\Lambda 0$

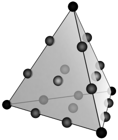
Elements $T \in \mathcal{T}_h$ are simplices in $\mathbb{R}^n$.

Shape fns: $V(T) = \mathcal{P}_r(T) = \mathcal{P}_r\Lambda^0(T)$ for some $r \geq 1$.

DOFs:

- $v \in \Delta_0(T)$: $\quad u \mapsto u(v)$
- $e \in \Delta_1(T)$: $\quad u \mapsto \int_e (\mathrm{tr}_e\, u) q, \quad q \in \mathcal{P}_{r-2}(e)$
- $f \in \Delta_2(T)$: $\quad u \mapsto \int_f (\mathrm{tr}_f\, u) q, \quad q \in \mathcal{P}_{r-3}(f)$

  $\vdots$



$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r-d-1}\Lambda^d(f),\ f \in \Delta_d(T),\ d \geq 0$$

Theorem

The number of DOFs $= \dim \mathcal{P}_r(T)$ *and they are unisolvent. The imposed continuity exactly forces inclusion in* H^1.

Unisolvence for Lagrange elements in n dimensions

Shape fns: $V(T) = \mathcal{P}_r(T)$, DOFs: $u \mapsto \int_f (\operatorname{tr}_f u)q$, $q \in \mathcal{P}_{r-d-1}(f)$, $d = \dim f$

$\#\Delta_d(T)$ $\dim \mathcal{P}_{r-d-1}(f_d)$ $\dim \mathcal{P}_r(T)$

DOF count:

$$\#\text{DOF} = \sum_{d=0}^{n} \binom{n+1}{d+1}\binom{r-1}{d} = \binom{r+n}{n} = \dim \mathcal{P}_r(T).$$

Unisolvence proved by induction on dimension ($n = 1$ is obvious).

Suppose $u \in \mathcal{P}_r(T)$ and all DOFs vanish. Let f be a facet of T. Note

- $\operatorname{tr}_f u \in \mathcal{P}_r(f)$
- the DOFs associated to f and its subfaces applied to u coincide with the Lagrange DOFs in $\mathcal{P}_r(f)$ applied to $\operatorname{tr}_f u$

Therefore $\operatorname{tr}_f u$ vanishes by the inductive hypothesis. Thus $u = (\prod_{i=0}^{n} \lambda_i)p$, $p \in \mathcal{P}_{r-n-1}(T)$. Choose $q = p$ in the interior DOFs to see that $p = 0$.

Polynomial differential forms

- Polynomial diff. forms: $\mathcal{P}_r\Lambda^k(\Omega)$ $\sum_\sigma a_\sigma\, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}$, $a_\sigma \in \mathcal{P}_r(\Omega)$

  Homogeneous polynomial diff. forms: $\mathcal{H}_r\Lambda^k(\Omega)$

- $$\dim \mathcal{P}_r\Lambda^k = \binom{r+n}{r}\binom{n}{k} = \binom{r+n}{r+k}\binom{r+k}{k}$$

  $$\dim \mathcal{H}_r\Lambda^k = \binom{r+n-1}{r}\binom{n}{k} = \frac{n}{n+r}\binom{r+n}{r+k}\binom{r+k}{k}$$

- *(Homogeneous) polynomial de Rham subcomplex:*

$$0 \longrightarrow \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \longrightarrow 0$$

$$0 \longrightarrow \mathcal{H}_r\Lambda^0 \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{H}_{r-n}\Lambda^n \longrightarrow 0$$

The Koszul complex

For $x \in \Omega \subset \mathbb{R}^n$, $T_x\Omega$ may be identified with $\mathbb{R}^n$, so the identity map can be viewed as a vector field.

- The Koszul differential $\kappa : \Lambda^k \to \Lambda^{k-1}$ is the contraction with the identity: $\kappa\omega = \omega \lrcorner \mathrm{id}$. Applied to polynomials it increases degree.
- $\kappa \circ \kappa = 0$ giving the *Koszul complex:*

$$0 \longrightarrow \mathcal{P}_r\Lambda^n \xrightarrow{\kappa} \mathcal{P}_{r+1}\Lambda^{n-1} \xrightarrow{\kappa} \cdots \mathcal{P}_{r+n}\Lambda^0 \longrightarrow 0$$

- $\kappa\, dx^i = x^i, \qquad \kappa(\omega \wedge \mu) = (\kappa\omega) \wedge \mu \pm \omega \wedge (\kappa\mu)$
- $\kappa(f\, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}) = f \sum_{i=1}^k (-)^i x^{\sigma_i}\, dx^{\sigma_1} \wedge \cdots \widehat{dx^{\sigma_i}} \cdots \wedge dx^{\sigma_k}$
- 3D Koszul complex:

$$0 \longrightarrow \mathcal{P}_r\Lambda^3 \xrightarrow{x} \mathcal{P}_{r+1}\Lambda^2 \xrightarrow{\times x} \mathcal{P}_{r+2}\Lambda^1 \xrightarrow{\cdot x} \mathcal{P}_{r+3}\Lambda^0 \longrightarrow 0$$

THEOREM (HOMOTOPY FORMULA)

$$(d\kappa + \kappa d)\omega = (r+k)\omega, \quad \omega \in \mathcal{H}_r\Lambda^k.$$

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Proof by induction on k. $k = 0$ is Euler's identity.
Assume true for $\omega \in \mathcal{H}_r\Lambda^{k-1}$, and verify it for $\omega \wedge dx^i$.

$$\begin{aligned} d\kappa(\omega \wedge dx^i) &= d(\kappa\omega \wedge dx^i + (-1)^{k-1}\omega \wedge x^i) \\ &= d(\kappa\omega) \wedge dx^i + (-1)^{k-1}(d\omega) \wedge x^i + \omega \wedge dx^i. \end{aligned}$$

$$\kappa d(\omega \wedge dx^i) = \kappa(d\omega \wedge dx^i) = \kappa(d\omega) \wedge dx^i + (-1)^k d\omega \wedge x^i.$$

$$(d\kappa + \kappa d)(\omega \wedge dx^i) = [(d\kappa + \kappa d)\omega] \wedge dx^i + \omega \wedge dx^i = (r+k)(\omega \wedge dx^i).$$

Consequences of the homotopy formula

- The polynomial de Rham complex is exact (except for constant 0-forms in the kernel). The Koszul complex is exact (except for constant 0-forms in the coimage).
- $\kappa d\omega = 0 \implies d\omega = 0, \quad d\kappa\omega = 0 \implies \kappa\omega = 0$
- $\mathcal{H}_r\Lambda^k = \kappa\mathcal{H}_{r-1}\Lambda^{k+1} \oplus d\mathcal{H}_{r+1}\Lambda^{k-1}$
- Define $\mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$
- $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0, \quad \mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n, \quad$ else $\mathcal{P}_{r-1}\Lambda^k \subsetneq \mathcal{P}_r^-\Lambda^k \subsetneq \mathcal{P}_r\Lambda^k$
- $\dim \mathcal{P}_r^-\Lambda^k = \binom{r+n}{r+k}\binom{r+k-1}{k} = \frac{r}{r+k}\dim \mathcal{P}_r\Lambda^k$
- $\mathcal{R}(d|\mathcal{P}_r^-\Lambda^k) = \mathcal{R}(d|\mathcal{P}_r\Lambda^k), \quad \mathcal{N}(d|\mathcal{P}_r^-\Lambda^k) = \mathcal{N}(d|\mathcal{P}_{r-1}\Lambda^k)$
- The complex (with constant r)

$$0 \to \mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n \to 0$$

  is exact (except for constant 0-forms).

On an n-D domain there are 2^{n-1} complexes beginning with $\mathcal{P}_r\Lambda^0$ (or ending with $\mathcal{P}_r\Lambda^n$). At each step we have two choices:

$$\mathcal{P}_r\Lambda^{k-1} \nearrow \mathcal{P}_r^-\Lambda^k,\ \searrow \mathcal{P}_{r-1}\Lambda^k \quad \text{or} \quad \mathcal{P}_r^-\Lambda^{k-1} \nearrow \mathcal{P}_r^-\Lambda^k,\ \searrow \mathcal{P}_{r-1}\Lambda^k$$

In 3-D:

$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \mathcal{P}_r^-\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^3 \to 0.$$

$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-2}\Lambda^3 \to 0,$$

$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \mathcal{P}_{r-1}^-\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-2}\Lambda^3 \to 0,$$

$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \mathcal{P}_{r-2}\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-3}\Lambda^3 \to 0,$$

The $P_r^-\Lambda^k$ family of simplicial FE differential forms

Given: a mesh $\mathcal{T}_h$ of simplices T, $r \geq 1$, $0 \leq k \leq n$, we define $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ via:

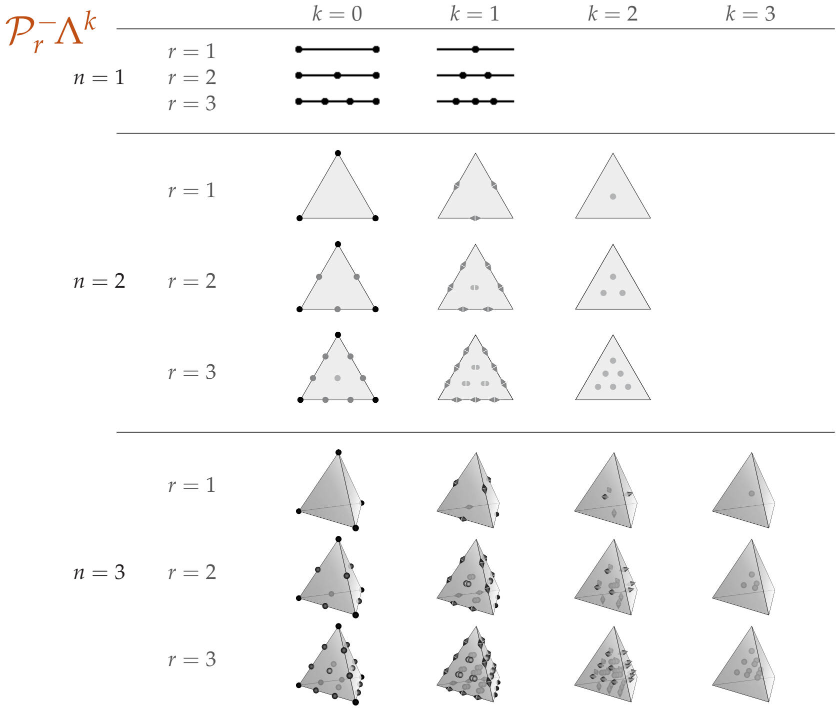
Shape fns: $\mathcal{P}_r^-\Lambda^k(T)$

DOFs:

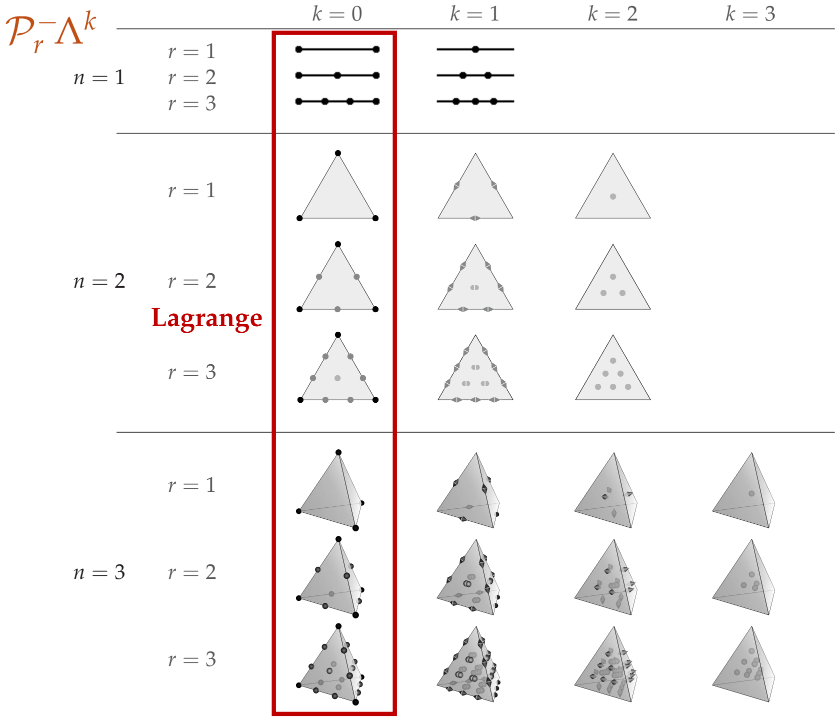
$$u \mapsto \int_f (\operatorname{tr}_f u)\wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f),\ f \in \Delta(T), \quad d = \dim f \geq k$$

Theorem

The number of DOFs $= \dim \mathcal{P}_r^-\Lambda^k(T)$ *and they are unisolvent. The imposed continuity exactly enforces inclusion in* $H\Lambda^k$.



$\mathcal{P}_r^-\Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$



$\mathcal{P}_r^- \Lambda^k$
$k = 0$
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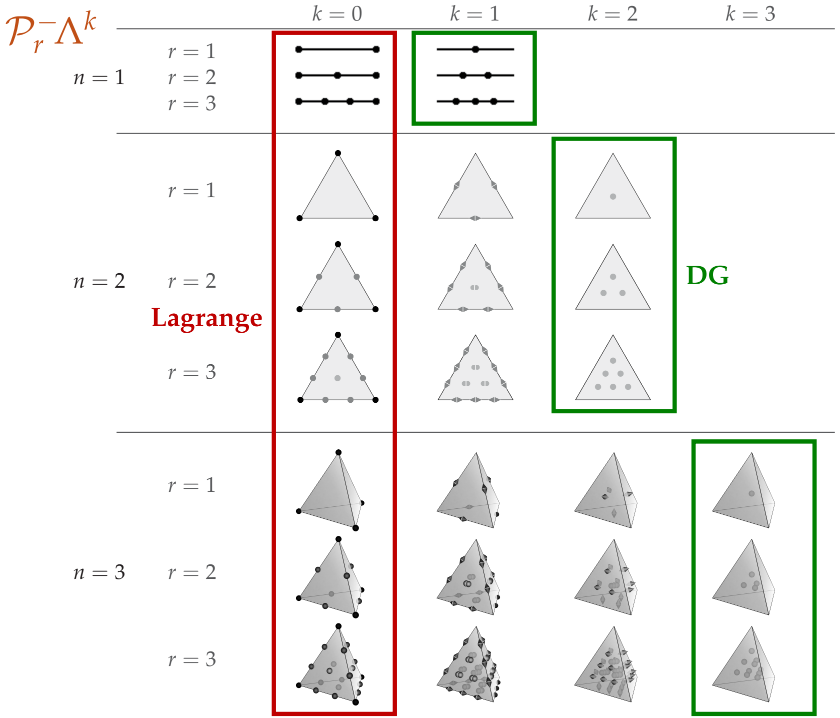
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$
Lagrange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$k = 2$
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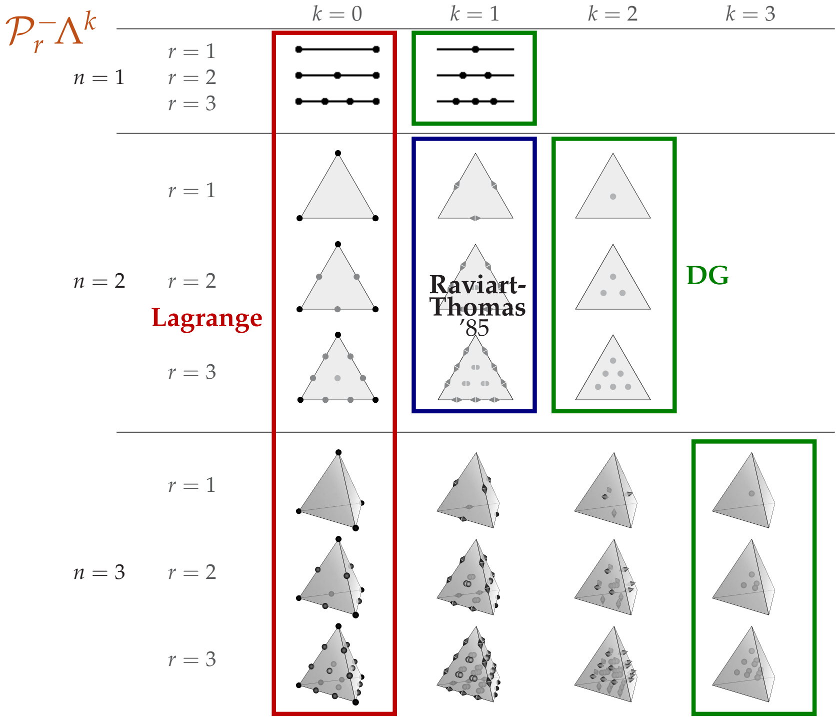
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$
Lagrange
DG



$\mathcal{P}_r^-\Lambda^k$
$k=0$
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$k=2$
$k=3$
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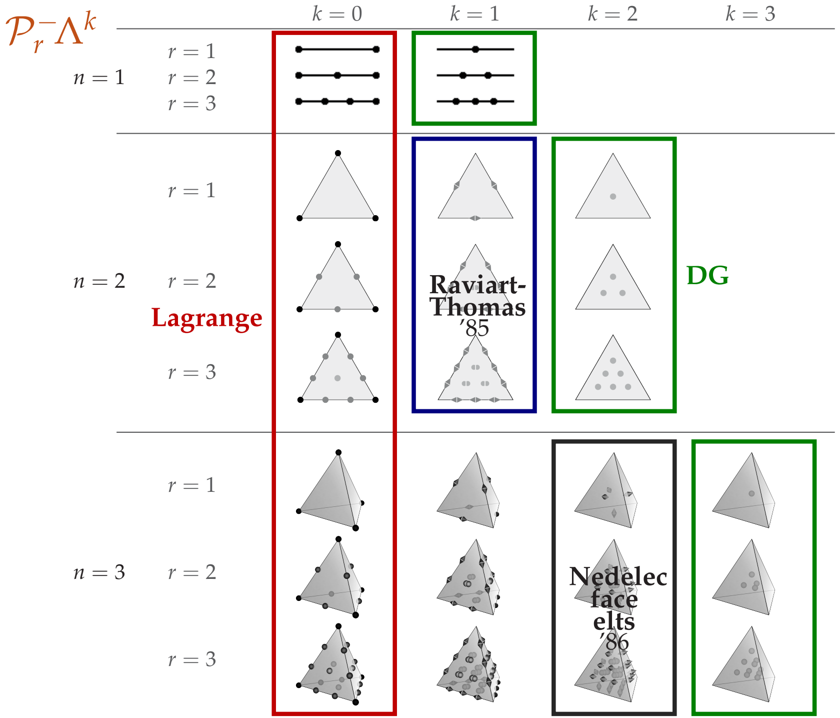
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$
Lagrange
Raviart-Thomas '85
DG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$n = 3$
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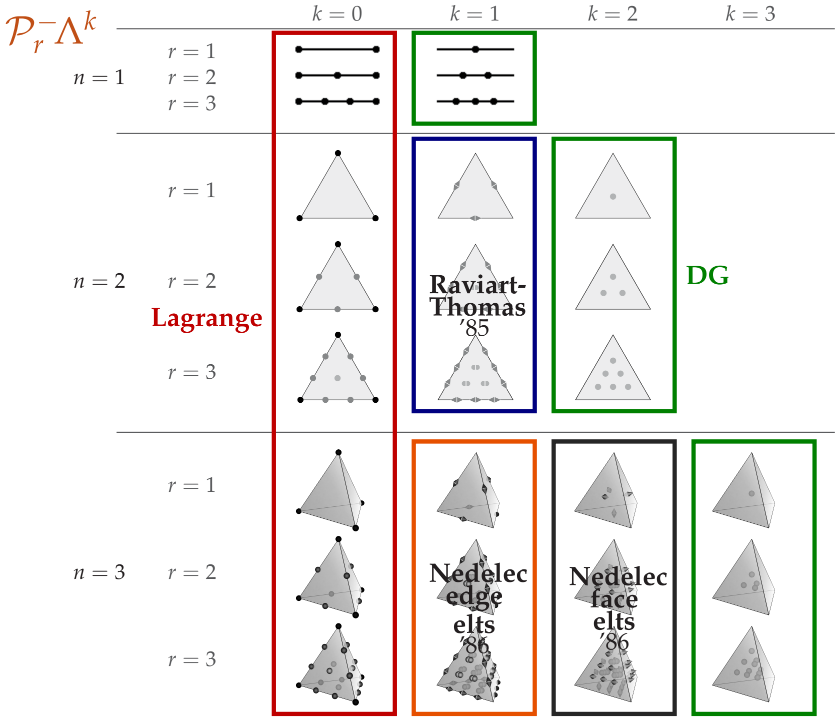
$r = 3$
Lagrange
Raviart-Thomas '85
DG
Nedelec face elts '86


$\mathcal{P}_r^-\Lambda^k$
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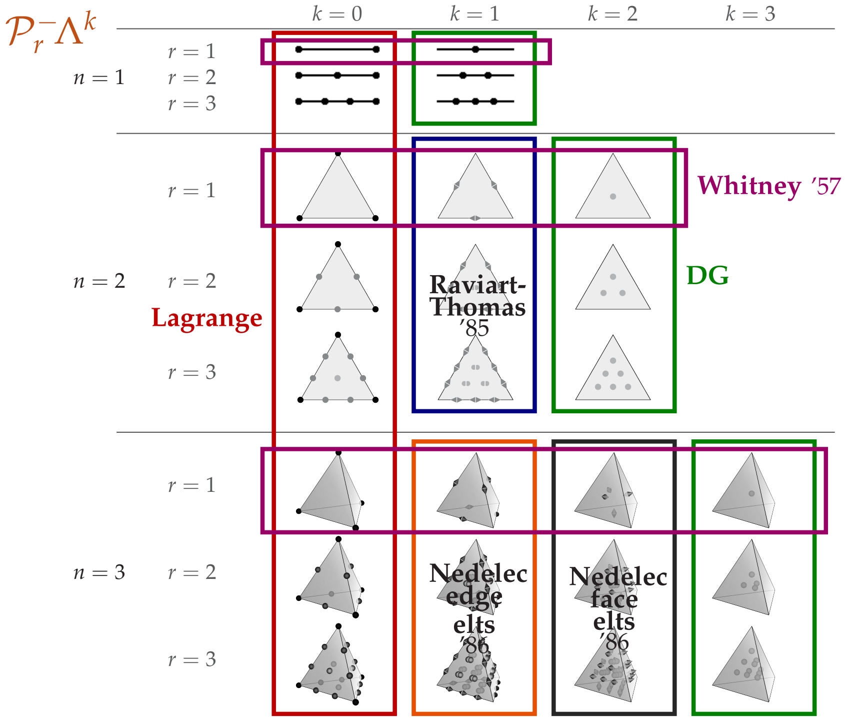
$r=3$
Lagrange
Raviart-Thomas '85
DG
Nedelec edge elts '86
Nedelec face elts '86



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$n=3$
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$r=3$
Whitney '57
DG
Lagrange
Raviart-Thomas '85
Nedelec edge elts '86
Nedelec face elts '86

Unisolvence for $\mathcal{P}_r^- \Lambda^k$: outline

1. Verify that the number of DOFs equals $\dim \mathcal{P}_r^- \Lambda^k(T)$

2. Verify the *trace properties:*

   a) $\operatorname{tr}_f \mathcal{P}_r^- \Lambda^k(T) \subset \mathcal{P}_r^- \Lambda^k(f)$, and

   b) the pullback $\operatorname{tr}_f^* : \mathcal{P}_r^- \Lambda^k(f)^* \to \mathcal{P}_r^- \Lambda^k(T)^*$ takes DOFs for $\mathcal{P}_r^- \Lambda^k(f)$ to DOFs for $\mathcal{P}_r^- \Lambda^k(T)$

3. $u \in \mathring{\mathcal{P}}_r^- \Lambda^k(T)$ & the interior DOFs vanish $\implies u = 0$

subspace w/ vanishing trace

1,2,3 $\implies$ unisolvence, by induction on dimension

Unisolvence for $\mathcal{P}_r^- \Lambda^k$: dimension count

$$
\begin{aligned}
\#\text{DOFs} &= \sum_{d \ge k} \#\Delta_d(T) \dim \mathcal{P}_{r+k-d-1}\Lambda^k(\mathbb{R}^d) \\
&= \sum_{d \ge k} \binom{n+1}{d+1} \binom{r+k-1}{d} \binom{d}{k} \\
&= \sum_{j \ge 0} \binom{n+1}{j+k+1} \binom{r+k-1}{j+k} \binom{j+k}{j}
\end{aligned}
$$

Simplify using the identities

$$
\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{a-b} \qquad \sum_{j \ge 0} \binom{a}{b+j}\binom{c}{j} = \binom{a+c}{a-b}
$$

to get

$$
\#\text{DOFs} = \binom{r+n}{r+k}\binom{r+k-1}{k} = \dim \mathcal{P}_r^- \Lambda^k
$$

Unisolvence for $\mathcal{P}_r^- \Lambda^k$, completed (modulo lemma)

2. The trace properties follows from definitions (essentially, $\operatorname{tr}_f \kappa u = \kappa_f \operatorname{tr}_u$).

3. It remains to show:

$$(\dagger)\ u \in \mathring{\mathcal{P}}_r^- \Lambda^k(T) \quad \& \quad (*)\ \int_T u \wedge q = 0\ \forall q \in \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T) \quad \Longrightarrow \quad u = 0$$

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A weaker result can be proven by an *explicit choice of test functions:*
Lemma:

$$(\ddagger)\ u \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(T) \quad \& \quad (*)\ \int_T u \wedge q = 0\ \forall q \in \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T) \quad \Longrightarrow \quad u = 0$$

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So we only need to show that $u \in \mathcal{P}_{r-1}\Lambda^k(T)$.

By the homotopy formula, $u \in \mathcal{P}_r^-\Lambda^k, du = 0 \implies u \in \mathcal{P}_{r-1}\Lambda^k$,

Unisolvence for $\mathcal{P}_r^- \Lambda^k$, completed (modulo lemma)

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By the homotopy formula, $u \in \mathcal{P}_r^- \Lambda^k$, $du = 0 \implies u \in \mathcal{P}_{r-1}\Lambda^k$, so it suffices to show that $du = 0$.

Unisolvence for $\mathcal{P}_r^-\Lambda^k$, completed (modulo lemma)

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So we only need to show that $u \in \mathcal{P}_{r-1}\Lambda^k(T)$.

By the homotopy formula, $u \in \mathcal{P}_r^-\Lambda^k, du = 0 \implies u \in \mathcal{P}_{r-1}\Lambda^k$, so it suffices to show that $du = 0$.

But $du \in \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(T)$ so satisfies ($\ddagger$) with $k \to k+1$. The hypothesis ($*$) for du then becomes: ($*$) $\int_T du\wedge q = 0\ \forall q \in \mathcal{P}_{r+k-n}\Lambda^{n-k-1}(T)$

This holds by integration by parts and ($*$).

Lemma

If $u \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(T)$ *and* $\int_T u \wedge q = 0, \quad q \in \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T)$ *then* $u \equiv 0$.

$$u = \sum_{\sigma \in \Sigma(k,n)} u_\sigma d\lambda_{\sigma_1} \wedge \cdots \wedge d\lambda_{\sigma_k}, \quad u_\sigma \in \mathcal{P}_{r-1}(T).$$

From the vanishing traces,

$$u_\sigma = p_\sigma \lambda_{\sigma_1^*} \cdots \lambda_{\sigma_{n-k}^*} \text{ for some } p_\sigma \in \mathcal{P}_{r+k-n-1}(T).$$

Choosing

$$q = \sum_{\sigma \in \Sigma(k,n)} (-1)^{\mathrm{sign}(\sigma,\sigma^*)} p_\sigma d\lambda_{\sigma_1^*} \wedge \cdots \wedge d\lambda_{\sigma_{n-k}^*}$$

gives

$$0 = \int_T u \wedge q = \int_T \sum_{\sigma \in \Sigma(k,n)} p_\sigma^2 \lambda_{\sigma_1^*} \cdots \lambda_{\sigma_{n-k}^*} d\lambda_1 \wedge \cdots \wedge d\lambda_n.$$

so all the p_σ vanish.

Summary for simplicial elements

The argument adapts easily to $\mathcal{P}_r\Lambda^k$. Thus a single argument proves unisolvence for all of the most important simplicial FE spaces at once.

To obtain the "best" proof, it is necessary

- to consider $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ together
- to consider all form degrees k
- to consider general dimension n

"A finite element which does not work in n-dimensions is probably not so good in 2 or 3 dimensions."

The $P_r\Lambda^k$ family of simplicial FE differential forms

Given: a mesh $\mathcal{T}_h$ of simplices T, $r \geq 1$, $0 \leq k \leq n$, we define $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ via:

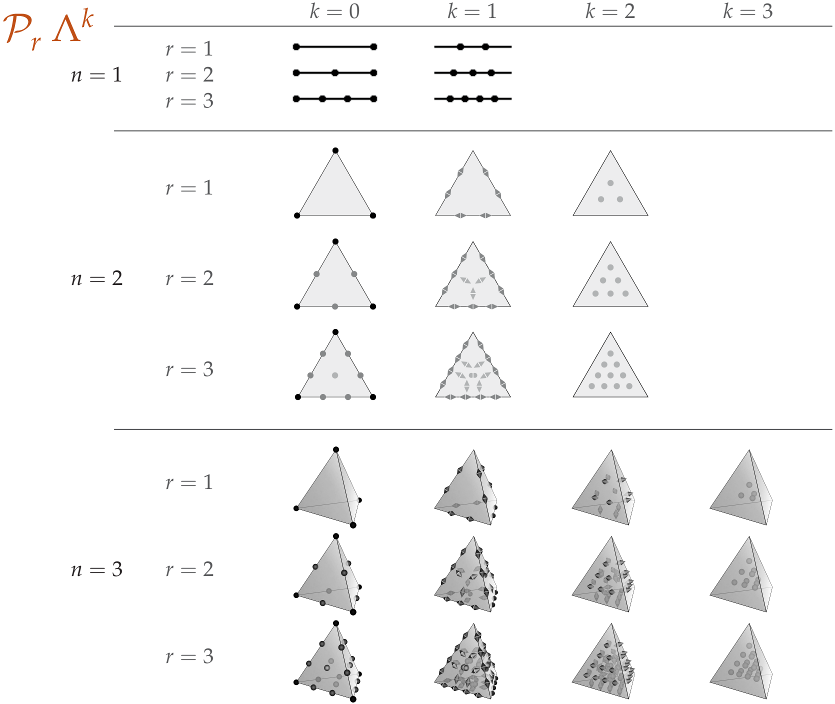
Shape fns: $\mathcal{P}_r\Lambda^k(T)$

DOFs:

$$u \mapsto \int_f (\operatorname{tr}_f u) \wedge q, \quad q \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f),\, f \in \Delta(T), \quad d = \dim f \geq k$$

Theorem

The number of DOFs $= \dim \mathcal{P}_r\Lambda^k(T)$ *and they are unisolvent. The imposed continuity exactly enforces inclusion in* $H\Lambda^k$*.*



$\mathcal{P}_r \Lambda^k$
$k = 0$
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$k = 2$
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$r = 1$
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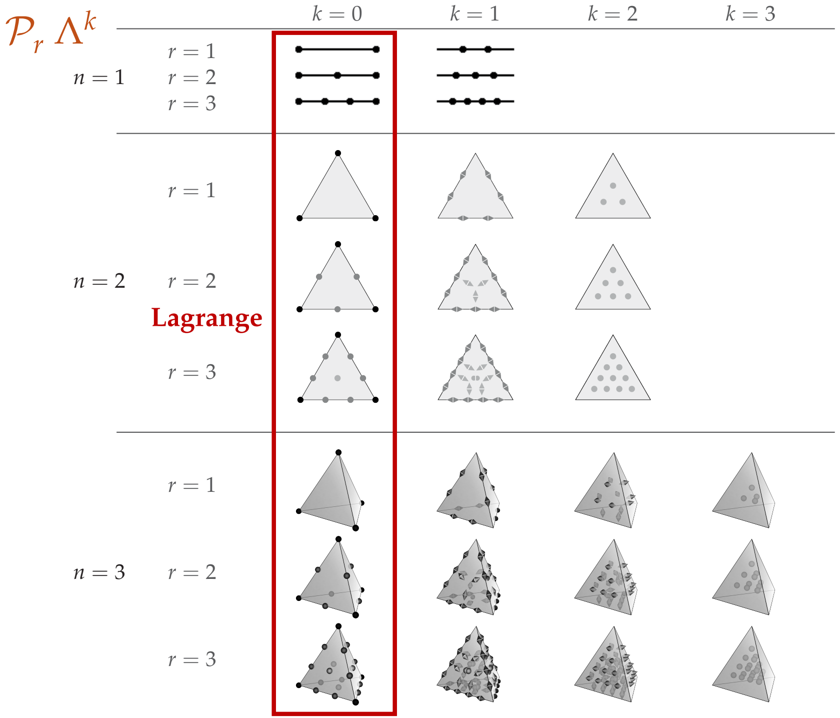

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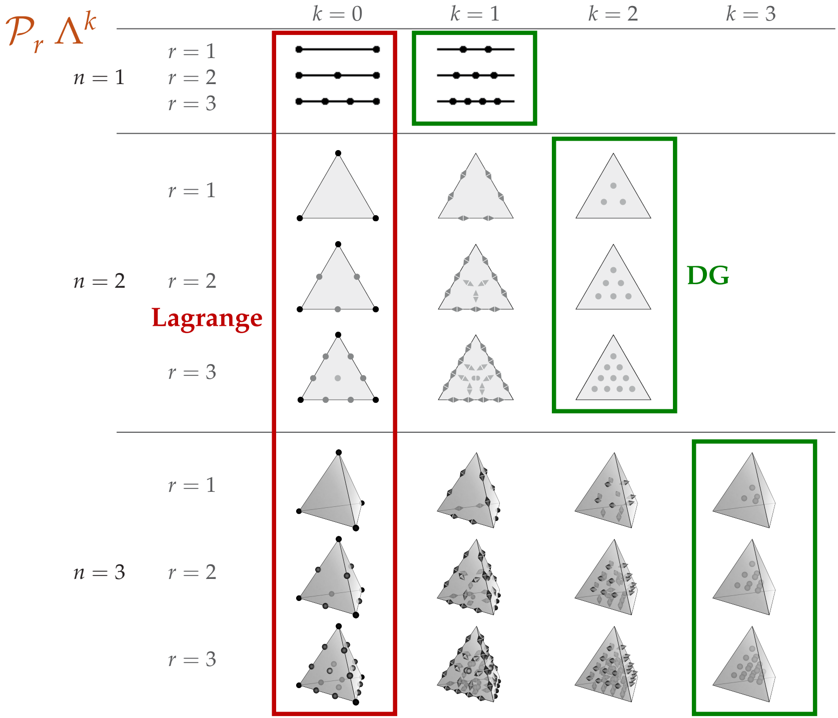
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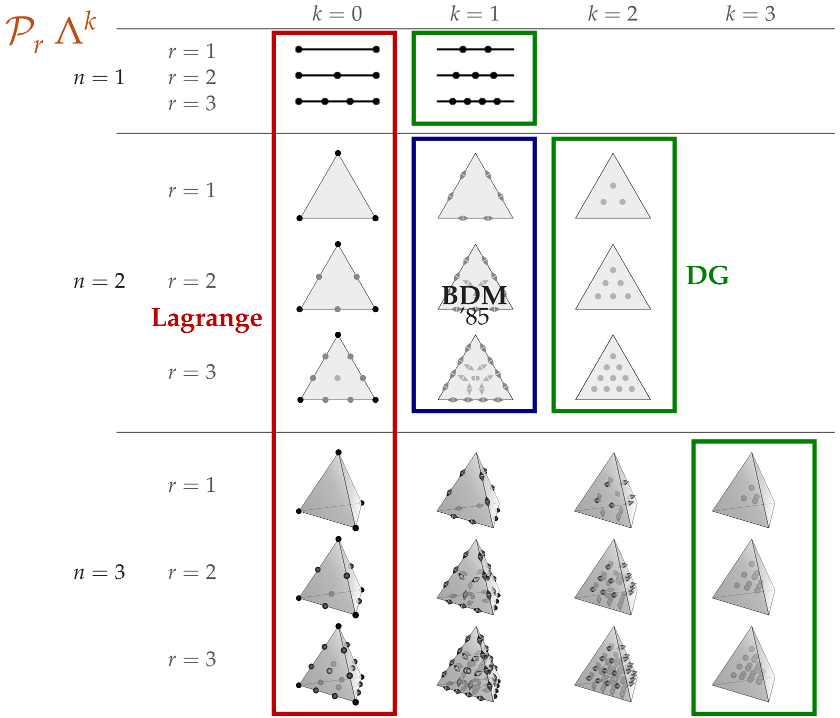
Lagrange
DG



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Lagrange
BDM
'85
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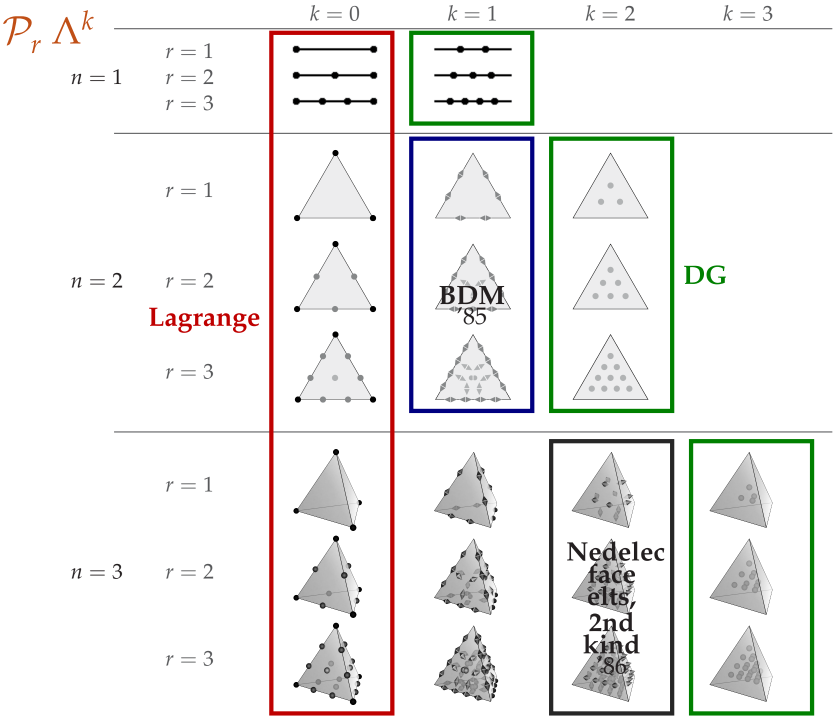

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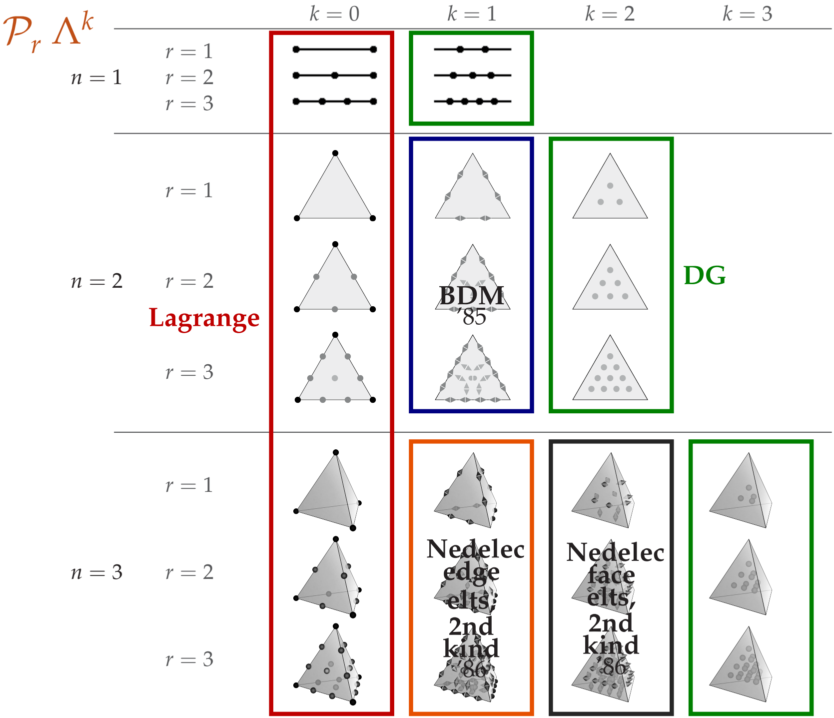
Lagrange
BDM '85
DG
Nedelec face elts, 2nd kind '86



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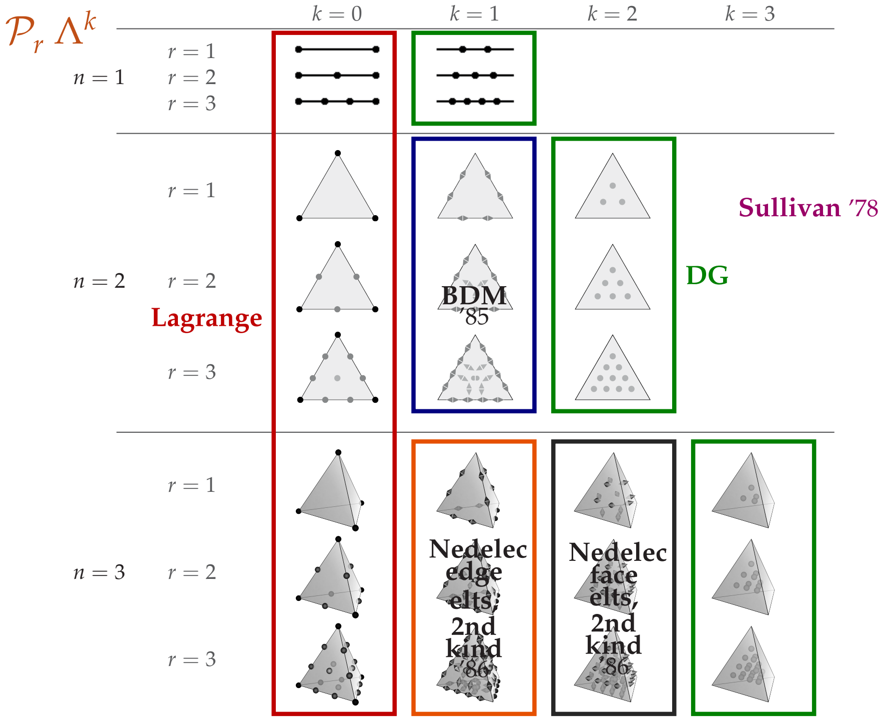

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$r=3$
Lagrange
BDM '85
DG
Sullivan '78
Nedelec edge elts, 2nd kind '86
Nedelec face elts, 2nd kind '86

Application of the $\mathcal{P}_r$ and $\mathcal{P}_r^-$ families to the Hodge Laplacian

- The shape function spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ combine into de Rham subcomplexes.
- The DOFs connect these spaces across elements to create subspaces of $H\Lambda^k(\Omega)$.

Therefore the assembled finite element spaces $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ combine into de Rham subcomplexes (in 2^{n-1} ways).

The DOFs of freedom determine projections from $\Lambda^k(\Omega)$ into the finite element spaces. From Stokes thm, these commute with d. Suitably modified, we obtain *bounded* cochain projections. Thus the abstract theory applies. We may use any two adjacent spaces in any of the complexes.

$$\left\{\begin{matrix} \mathcal{P}_r\Lambda^{k-1}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}) \end{matrix}\right\} \xrightarrow{d} \left\{\begin{matrix} \mathcal{P}_r^-\Lambda^{k}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r-1}\Lambda^{k}(\mathcal{T}) \end{matrix}\right\}$$

Rates of convergence

Rates of convergence are determined by the improved error estimates from the abstract theory. They depend on

- The smoothness of the data f.
- The amount of elliptic regularity.
- The degree of of complete polynomials contained in the finite element spaces.

The theory delivers the best possible results: with sufficiently smooth data and elliptic regularity, the rate of convergence for each of the quantities u, du, σ, $d\sigma$, and p in the L^2 norm is the best possible given the degree of polynomials used for that quantity.

Eigenvalues converge as $O(h^{2r})$.

Historical notes

- The $\mathcal{P}_1^-\Lambda^k$ complex is in Whitney '57 (Bossavit '88).
- In '76, Dodziuk and Patodi defined a finite difference approximation based on the Whitney forms to compute the eigenvalues of the Hodge Laplacian, and proved convergence. In retrospect, that method can be better viewed as a mixed finite element method. This was a step on the way to proving the Ray-Singer conjecture, completed in '78 by W. Miller.
- The $\mathcal{P}_r\Lambda^k$ complex is in Sullivan '78.
- Hiptmair gave a uniform treatment of the $\mathcal{P}_r^-\Lambda^k$ spaces in '99.
- The unified treatment and use of the Koszul complex is from DNA-Falk-Winther '06.

Bounded cochain projections

The DOFs defining $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ determine canonical projection operators Π_h from piecewise smooth forms in $H\Lambda^k$ onto Λ_h^k. However, Π_h is *not bounded* on $H\Lambda^k$ (much less uniformly bounded wrt h). Π_h *is* bounded on $C\Lambda^k$.

If we have a smoothing operator $R_{\epsilon,h} \in \mathrm{Lin}(L^2\Lambda^k, C\Lambda^k)$ such that $R_{\epsilon,h}$ commutes with d, we can define $Q_{\epsilon,h} = \Pi_h R_{\epsilon,h}$ and obtain a bounded operator $L^2\Lambda^k \to \Lambda_h^k$ which commutes with d (as suggested by Christiansen).

However Q_h will not be a projection. We correct this by using Schöberl's trick: if the finite dimensional operator

$$Q_{\epsilon,h}|_{\Lambda_h^k} : \Lambda_h^k \to \Lambda_h^k$$

is invertible, then

$$\pi_h := (Q_{\epsilon,h}|_{\Lambda_h^k})^{-1} Q_{\epsilon,h},$$

is a bounded commuting projection. It remains to get uniform bds on π_h.

The two key estimates

For this we need two key estimates for $Q_{\epsilon,h} := \Pi_h R_{\epsilon,h}$:

- For fixed ϵ, $Q_{\epsilon,h}$ is uniformly bounded:
  $\forall \epsilon > 0$ suff. small $\exists\, c(\epsilon) > 0$ s.t.

$$\sup_h \|Q_{\epsilon,h}\|_{\mathrm{Lin}(L^2,L^2)} \leq c(\epsilon)$$

- $\lim_{\epsilon \to 0} \|I - Q_{\epsilon,h}\|_{\mathrm{Lin}(L^2,L^2)} = 0 \quad$ uniformly in h

THEOREM

Suppose that these two estimates hold and define $\pi_h := (Q_{\epsilon,h}|_{\Lambda_h^k})^{-1} Q_{\epsilon,h}$, where Λ_h^k is either $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $\mathcal{P}_{r+1}^-\Lambda^k(T_h)$. Then, for h sufficiently small, π_h is a cochain projection onto Λ_h^k and

$$\|\omega - \pi_h \omega\| \leq c h^s \|\omega\|_{H^s\Lambda^k}, \quad \omega \in H^s\Lambda^k, \quad 0 \leq s \leq r+1.$$

The smoothing operator

The simplest definition is to take $R_{\epsilon,h}u$ to be an average over $y \in B_1$ of $(F^y_{\epsilon,h})^* u$ where $F^y_{\epsilon,h}(x) = x + \epsilon h y$:

$$R_{\epsilon,h}u(x) = \int_{B_1} \rho(y)[(F^y_{eh})^* u](x)\, dy$$

Needs modification near the boundary and for non-quasiuniform meshes.

The key estimates can be proven using macroelements and scaling.