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# ASYMPTOTIC CONVERGENCE RATES FOR THE KIRCHHOFF PLATE MODEL

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## ABSTRACT

In this thesis, we study the asymptotic convergence of the Kirchhoff plate model as an approximation to the full system of three-dimensional linear elasticity, considering the cases of soft and of hard simply supported boundary conditions, and the case of a periodic plate. Specifically we obtain the order of convergence of the energy norm of the differences between the exact three-dimensional stress and displacement fields and approximations to them obtained from the Kirchhoff solution.

We develop a new method of analysis that combines the existing variational energy method, singular perturbation techniques, and Saint Venant's principle. By using this method, we prove that for the hard simply supported plate with smooth boundary the known global convergence rate of $O(t^{1/2})$ is sharp. (This is the rate of convergence of the relative energy norm error.) We also show that when consideration is restricted to an interior domain, disjoint from the lateral boundary of the plate, the relative energy norm convergence rate for the hard simply supported plate increases to $O(t)$. When the same analysis is applied to the soft simply supported plate, both the global and interior convergence rates are found to be $O(t^{1/2})$. The analysis suggests, but does not establish, that these rates are sharp. These low orders of convergence are in contrast to case of the periodic plate where we show that second order convergence holds.

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# Chapter One

# INTRODUCTION

Consider the problem of finding the displacement and stress fields in a three-dimensional plate which result from loads applied to its top and bottom surfaces. The theory of linearized elasticity theory determines these fields as the solution of a boundary value problem posed over the three-dimensional domain. When the plate is thin, two-dimensional plate models are often used to approximate the three-dimensional problem. This approach is known as dimensional reduction. The most popular such model is the Kirchhoff plate model, which determines a scalar field on the midsection of the plate as the solution to a biharmonic problem. One can then compute approximations to the three-dimensional displacement and stress fields on the entire plate from this scalar field. In this thesis we consider the accuracy of these approximations. Specifically we consider the order with which the error tends to zero as the plate thickness tends to zero.

Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain in the plane and $t > 0$. We suppose that the plate occupies the three-dimensional domain

$$P^t = \Omega \times (-1/2, 1/2).$$

We denote the top and bottom surfaces $\Omega \times \{t/2\}$ and $\Omega \times \{-t/2\}$ by $\Omega^t_+$ and $\Omega^t_-$ respectively, and the lateral surface $\partial\Omega \times (-t/2, t/2)$ by $\Gamma^t$. The linearized equations of elasticity then require that the displacement field $\vec{u}$ and the stress field $\boldsymbol{\sigma}$ satisfy the differential equations

$$\boldsymbol{\sigma} = \frac{E}{1+\nu}\left[\boldsymbol{\varepsilon}(\vec{u}) + \frac{\nu}{1-2\nu}(\operatorname{div}\vec{u})\,\boldsymbol{\delta}\right] \qquad \text{in } P^t, \tag{1.1}$$

$$\vec{\operatorname{div}}\boldsymbol{\sigma} = \vec{0} \qquad \text{in } P^t, \tag{1.2}$$

where $E$ is the Young's modulus, $\nu$ the Poisson ratio, and $\boldsymbol{\delta}$ the $3\times 3$ identity matrix. The surface loading is specified by the boundary conditions

$$\boldsymbol{\sigma}\vec{n} = (0,0,q_+)^T \quad \text{on } \Omega_+^t, \qquad \boldsymbol{\sigma}\vec{n} = (0,0,q_-)^T \quad \text{on } \Omega_-^t, \tag{1.3}$$

where $q_\pm : \Omega_\pm \to \mathbb{R}$ are given. We shall only consider the case where the top and bottom surface loads are equal. Moreover, for convenience, we assume that these are scaled to be proportional to $t^3$. (This does not lead to any loss of generality, since the problem is linear and we can simply adapt our results to other scalings.) Thus we assume that

$$q_+(x,y,t/2) = \frac{t^3}{2}g(x,y), \quad q_-(x,y,t/2) = \frac{t^3}{2}g(x,y) \qquad \text{for all } (x,y)\in\Omega, \tag{1.4}$$

where $g : \Omega \to \mathbb{R}$ is a given smooth function.

We shall consider the case of a simply supported plate. Actually we shall investigate two different boundary conditions on the lateral boundary $\Gamma^t$ which model this situation. To describe these boundary conditions we introduce the coordinate directions $\vec{e}_3$, $\vec{n}$, and $\vec{s}$ at each point of the the lateral boundary. The *soft simply supported* plate satisfies

$$\vec{u}\cdot\vec{e}_3 = \vec{s}^T\boldsymbol{\sigma}\vec{n} = \vec{n}^T\boldsymbol{\sigma}\vec{n} = 0 \qquad \text{on } \Gamma^t. \tag{1.5}$$

The *hard simply supported plate* satisfies

$$\vec{u}\cdot\vec{e}_3 = \vec{u}\cdot\vec{s} = \vec{n}^T\boldsymbol{\sigma}\vec{n} = 0 \qquad \text{on } \Gamma^t. \tag{1.6}$$

Note that the soft simply supported plate is the more usual boundary condition.

Both the soft simply supported plate problem (1.1)–(1.3), (1.5) and the hard simply supported plate problem (1.1)–(1.3), (1.6) admit a solution $(\boldsymbol{\sigma}, \vec{u})$. For the hard simply supported plate, this solution is uniquely determined. For the soft simply supported plate, it is easy to derive from [9] that the solution is determined up to addition of an in-plane rigid motion, i.e., a function in the set

$$\mathcal{R} = \left\{ \vec{v}(x,y,z) = (a + cy, b - cx, 0) \,|\, a, b, c \in \mathbb{R} \right\}. \tag{1.7}$$

The solution is then rendered unique by imposing the side condition

$$\int_{P^t} \vec{u} \cdot \vec{r} = 0 \qquad \text{for all } \vec{r} \in \mathcal{R}. \tag{1.8}$$

For the problem just described the Kirchhoff plate model determines a function $w : \Omega \to \mathbb{R}$ by the biharmonic equation

$$\frac{E}{12(1-\nu^2)} \Delta^2 w = g \qquad \text{in } \Omega, \tag{1.9}$$

and the boundary conditions

$$w = 0, \qquad \underset{\sim}{n}^T \left[ (1-\nu) \underset{\approx}{\operatorname{grad}} (\underset{\sim}{\operatorname{grad}} w) + \nu \Delta w \, \underset{\approx}{\delta} \right] \underset{\sim}{n} = 0 \qquad \text{on } \partial\Omega. \tag{1.10}$$

Note that the same boundary conditions (1.10) are used to replace either (1.5) or (1.6): the distinction between soft and hard simply supported plates vanishes. From the solution $w$ to (1.9)–(1.10), we may construct approximations $\vec{u}^{\mathrm{k}}$ to $\vec{u}$ and $\boldsymbol{\sigma}^{\mathrm{k}}$ to $\boldsymbol{\sigma}$. For example,

$$\sigma_{11}^{\mathrm{k}} = -\frac{Ez}{1-\nu^2} \left[ (1-\nu) \frac{\partial^2 w}{\partial x^2} + \nu \Delta w \right].$$

Explicit expressions are given below in (2.2.2), (2.2.4), and (2.2.6).

In order to discuss the accuracy of the Kirchhoff approximation we introduce the energy norms for the displacement and for the stress, defined by

$$|||\boldsymbol{v}|||^2 = \int_{P^t} [A^{-1} \boldsymbol{\varepsilon}(\boldsymbol{v})] : \boldsymbol{\varepsilon}(\boldsymbol{v}), \qquad \|\boldsymbol{\tau}\|_E^2 = \int_{P^t} (A\boldsymbol{\tau}) : \boldsymbol{\tau}, \tag{1.11}$$

respectively, where

$$A\boldsymbol{\sigma} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\operatorname{tr}\boldsymbol{\sigma})\boldsymbol{\delta}, \tag{1.12}$$
$$\int_{P^t} \boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i,j=1}^{3} \int_{P^t} \sigma_{ij}\tau_{ij}.$$

Note that when $\boldsymbol{\tau} = A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{v})$, $|||\boldsymbol{v}||| = \|\boldsymbol{\tau}\|_E$.

The following theorem gives the basic global bound on the error in the Kirchhoff approximation.

**Theorem 1.1.** *Let* $\boldsymbol{\sigma}$ *and* $\vec{u}$ *be defined by either the soft simply supported plate problem* (1.1)–(1.4), (1.5), (1.8) *or the hard simply supported plate problem* (1.1)–(1.4), (1.6), *and let* $\boldsymbol{\sigma}^{\mathrm{k}}$ *and* $\vec{u}^{\mathrm{k}}$ *be the Kirchhoff approximations defined by* (1.9), (1.10), (2.2.2), *and* (2.2.6). *Then there exists a constant* $C$ *depending only on the domain* $\Omega$ *such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u} - \vec{u}^{\mathrm{k}}||| \le Ct^2\|g\|_{L^2(\Omega)}. \tag{1.13}$$

The interpretation of this theorem is not straightforward. While the constant $C$ and the function $g$ are independent of $t$, the three-dimensional solution $(\boldsymbol{\sigma}, \vec{u})$ depends on $t$, as do the energy norms $|||\cdot|||$ and $\|\cdot\|_E$. In fact, as we shall see, if $g$ does not vanish identically, then

$$c_1 t^{3/2} \le \|\boldsymbol{\sigma}\|_E = |||\vec{u}||| \le c_2 t^{3/2},$$

where $c_1$ and $c_2$ are positive constants depending on $\Omega$ and $g$, but independent of $t$. Therefore, from Theorem 1.1 we obtain the relative error estimate

$$\frac{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E}{\|\boldsymbol{\sigma}\|_E} + \frac{|||\vec{u} - \vec{u}^{\mathrm{k}}|||}{|||\vec{u}|||} \le C't^{1/2},$$

with $C'$ independent of $t$. Thus the Kirchhoff model gives an $O(t^{1/2})$ approximation of the three-dimensional solution, when measured in energy norm. This rather low rate of convergence is in fact sharp, at least for the hard simply supported plate. Indeed, as we show below, if $g$ does not vanish identically then

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u} - \vec{u}^{\mathrm{k}}||| \geq ct^2$$

with $c > 0$ independent of $t$. Although it does not follow from the analysis below, it seems very likely that the $O(t^{1/2})$ convergence rate is sharp for the soft simply supported plate as well.

The solution of the three-dimensional plate problem has a complex boundary layer when $t$ is small [4], [3], [2], [13], [17], [20], however the Kirchhoff approximation has no boundary layer whatever. This suggests that poor approximation near the lateral boundary may be responsible for the low rate of convergence in the energy norm, and the approximation may be more accurate away from the boundary. In fact this is true for the hard simply supported plate problem. More precisely, for $P_0^t \subset P^t$ define

$$|||\boldsymbol{v}|||_{P_0^t}^2 = \int_{P_0^t} [A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{v})] : \boldsymbol{\varepsilon}(\boldsymbol{v}), \qquad \|\boldsymbol{\tau}\|_{E,P_0^t}^2 = \int_{P_0^t} (A\boldsymbol{\tau}) : \boldsymbol{\tau}.$$

Then we have the following interior convergence theorem, proved as Theorem 3.9.5 below.

**Theorem 1.2.** *Let $\Omega_0$ satisfy $\bar{\Omega}_0 \subset \Omega$, and set $P_0^t = \Omega_0 \times (-t/2, t/2)$. Let $(\boldsymbol{\sigma}, \vec{u})$ and $(\boldsymbol{\sigma}^{\mathrm{k}}, \vec{u}^{\mathrm{k}})$ be as in Theorem 1.1. Then, in the case of the hard simply supported plate, there exists a constant $C$ depending only on the domain and $\Omega$ such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_{E,P_0^t} + |||\vec{u} - \vec{u}^{\mathrm{k}}|||_{P_0^t} \leq Ct^{5/2}\|g\|_{L^2(\Omega)}. \tag{1.14}$$

Thus for the hard simply supported plate, the Kirchhoff approximation converges with first order in the energy norm on subdomains bounded away from the lateral boundary. The proof of this theorem is one of the main results of the thesis.

As we shall see, the same analysis, when applied to the soft simply supported plate, only gives an interior convergence rate of $O(t^{1/2})$, i.e., no higher than the global rate.

Another indication that the boundary layer is in some sense responsible for the low order of convergence of the Kirchhoff model is obtained by considering a bi-periodic plate. In Chapter 2 we shall show that in this case the global estimate (1.13) can be improved to

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u} - \vec{u}^{\mathrm{k}}||| \le Ct^{7/2}\|g\|_{L^2(\Omega)},$$

i.e., for the periodic problem (for which there is no boundary layer), the relative energy error of Kirchhoff approximation is $O(t^2)$.

In 1959 Morgenstern [18] proved one of the first convergence results for the Kirchhoff model. For several boundary conditions, including soft simply supported, he showed that the relative global energy error tends to zero with the plate thickness. Although he did not discuss the convergence rate, it is not difficult to extend his arguments to obtain Theorem 1.1, that is, a relative energy error of $O(t^{1/2})$. Morgenstern used a variational approach based on the duality of the displacement energy and the complementary energy (essentially the Prager-Synge theorem). In 1971 Simmonds [21] used the same approach to show that under very special boundary condition, which he termed "regular," the relative energy error is $O(t^2)$. The bi-periodic plate may be viewed as a regular boundary value problem. However most other common boundary conditions, including both soft and hard simply supported plates (and clamped and free plates as well), are not regular in the sense of Simmonds. In 1990 Babuška and Pitkäranta [7] also employed the approach of Morgenstern. They showed that for the hard simply supported Kirchhoff plate, the convergence rate is $O(t^{1/2})$ for domains with smooth boundary and that this rate increases to $O(t)$ for polygonal domains. The first person to discuss interior convergence rates was Destuynder [11], who considered the global and interior energy convergence of clamped plate in his thesis. Destuynder's approach is based on singular perturbation techniques and Fourier analysis, and is quite different from Morgenstern's.

In this thesis we develop a new method of analysis combining the variational approach of Morgenstern and the singular perturbation techniques of Destuynder. Saint Venant's principle also plays an essential role in our approach. We believe that the resulting analysis gives sharper and clearer results. In particular, the proof of the sharpness of the global estimate for the hard simply supported plate and the interior estimate (1.14) are, to the best of our knowledge, new results. Our approach has some common features with Schwab's work on dimensional reduction of the Laplacian on a thin three-dimensional domain [19].

Although we discuss only the Kirchhoff plate in the thesis, the order analysis for the Kirchhoff plate may help in the study of other plate models. For example, the Reissner–Mindlin plate is the next simplest two-dimensional plate model, and is preferred in many applications. The convergence rates for the Reissner–Mindlin plate can be derived from those of the Kirchhoff plate by using Arnold and Falk's results in [2], [3], and [4]. In these papers, they found among other things the gap between the Kirchhoff plate solution and the Reissner–Mindlin plate solution. The convergence rates of the Reissner–Mindlin plate can thus be obtained by using a triangle inequality. In recent years, much work has been on the hierarchical two-dimensional plate models [1], [5], [20], [19], [6]. The approach we use in this thesis may also provide a way to explore the convergence rates for those higher order plate models.

# Chapter Two

## TOOLS AND METHODS

In this chapter, we introduce the Prager–Synge theorem and discuss its application in the plate convergence problems. The theorem provides a tool to find convergence rates without referring to the three-dimensional exact solution. It thus avoids the discussion of possible boundary layer complications. For a bi-periodic loading problem, the theorem can easily be applied to obtain an order $O(t^2)$ convergence rate. For the soft and hard simply supported plates, applications of the theorem to its full advantage is not straightforward. Methods for using the theorem effectively are discussed.

We write the following notational conventions throughout the thesis. Latin indices $i$ and $j$ generally range from 1 to 3 while Greek indices $\alpha$ and $\beta$ range from 1 to 2. Unless otherwise stated, Latin letters with superimposed arrows, such as $\vec{v}$, are used to denote vectors in $\mathbb{R}^3$ and bold Greek letters, such as $\boldsymbol{\tau}$, are used to denote $3 \times 3$ symmetric tensors. Tilde underscored Latin letters, like $\underset{\sim}{v}$ stand for vectors in $\mathbb{R}^2$ and double tilde underscored Greek letters, like $\underset{\approx}{\tau}$, for $2 \times 2$ symmetric tensors. When specifications are needed in the equations, to save space, we use soft case for the soft simply supported plate, and hard case for the hard simply supported plate. A complete list of notations is given at the end of the thesis.

## Sec 2.1. The Prager–Synge Theorem

To state the theorem, we first define spaces $\boldsymbol{\Sigma}$ and $\boldsymbol{V}$ by

$$\begin{aligned}
&\boldsymbol{\Sigma} = \left\{ \boldsymbol{\sigma} \,|\, \sigma_{ij} \in L^2(P^t), \sigma_{ij} = \sigma_{ji} \right\}, \\
&\boldsymbol{V} = \left\{ \vec{v} \,|\, v_i \in H^1(P^t), \vec{v} \cdot \vec{e}_3 = 0 \text{ and (1.8) holds} \right\} \quad \text{soft case,} \\
&\boldsymbol{V} = \left\{ \vec{v} \,|\, v_i \in H^1(P^t), \vec{v} \cdot \vec{e}_3 = \vec{v} \cdot \vec{s} = 0 \right\} \quad \text{hard case.}
\end{aligned} \tag{2.1.1}$$

The weak formulation for the three-dimensional plate is the following:

Find $(\boldsymbol{\sigma}, \vec{u}) \in \boldsymbol{\Sigma} \times \boldsymbol{V}$ such that

$$\begin{aligned}
&\int_{P^t} (A\boldsymbol{\sigma}) : \boldsymbol{\tau} + \int_{P^t} \boldsymbol{\varepsilon}(\vec{u}) : \boldsymbol{\tau} = 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \\
&\int_{P^t} \boldsymbol{\varepsilon}(\vec{v}) : \boldsymbol{\sigma} = \int_{\Omega^t_+} q_+ v_3 + \int_{\Omega^t_-} q_- v_3 && \text{for all } \vec{v} \in \boldsymbol{V}.
\end{aligned} \tag{2.1.2}$$

**Theorem 2.1.1 (Prager–Synge Theorem).** *Let* $(\boldsymbol{\sigma}, \vec{u})$ *be the solution to* (2.1.2). *Then for any* $\tilde{\vec{u}} \in \boldsymbol{V}$ *and any* $\tilde{\boldsymbol{\sigma}} \in \boldsymbol{\Sigma}$ *satisfying the following constraint*

$$\int_{P^t} \tilde{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}(\vec{v}) = \int_{\Omega^t_+} q_+ v_3 + \int_{\Omega^t_-} q_- v_3 \qquad \textit{for all } \vec{v} \in \boldsymbol{V}, \tag{2.1.3}$$

*the following identity holds:*

$$|||\vec{u} - \tilde{\vec{u}}|||^2 + \|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}\|_E^2 = \|\tilde{\boldsymbol{\sigma}} - A^{-1}\boldsymbol{\varepsilon}(\tilde{\vec{u}})\|_E^2. \tag{2.1.4}$$

*Proof.* Since $\boldsymbol{\sigma} = A^{-1}\boldsymbol{\varepsilon}(\vec{u})$ and $\vec{u} - \tilde{\vec{u}} \in \boldsymbol{V}$, then

$$\begin{aligned}
&\|\boldsymbol{\sigma} - [\tilde{\boldsymbol{\sigma}} + A^{-1}\boldsymbol{\varepsilon}(\tilde{\vec{u}})]/2\|_E^2 - \|[\tilde{\boldsymbol{\sigma}} - A^{-1}\boldsymbol{\varepsilon}(\tilde{\vec{u}})]/2\|_E^2 \\
&\quad = \int_{P^t} (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) A[\boldsymbol{\sigma} - A^{-1}\boldsymbol{\varepsilon}(\tilde{\vec{u}})] = \int_{P^t} (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) : \boldsymbol{\varepsilon}(\boldsymbol{u} - \tilde{\boldsymbol{u}}) \\
&\quad = \int_{\Omega^t_+} q_+(u_3 - \tilde{u}_3) - \int_{\Omega^t_+} q_+(u_3 - \tilde{u}_3) + \int_{\Omega^t_-} q_-(u_3 - \tilde{u}_3) - \int_{\Omega^t_-} q_-(u_3 - \tilde{u}_3) = 0.
\end{aligned}$$

It follows that

$$\|2\boldsymbol{\sigma} - (A^{-1}\boldsymbol{\varepsilon}(\tilde{\vec{u}}) + \tilde{\boldsymbol{\sigma}})\|_E^2 = \|\tilde{\boldsymbol{\sigma}} - A^{-1}\boldsymbol{\varepsilon}(\tilde{\vec{u}})\|_E^2.$$

On the other hand, we have also the following identities:

$$\|2\boldsymbol{\sigma} - [\tilde{\boldsymbol{\sigma}} + A^{-1}\boldsymbol{\varepsilon}(\tilde{\vec{u}})]\|_E^2 = \|\boldsymbol{\sigma} - A^{-1}\boldsymbol{\varepsilon}(\tilde{\vec{u}})\|_E^2 + \|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}\|_E^2$$
$$- 2\int_{P^t}(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}})A[\boldsymbol{\sigma} - A^{-1}\boldsymbol{\varepsilon}(\tilde{\vec{u}})] = |||\vec{u} - \tilde{\vec{u}}|||^2 + \|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}\|_E^2.$$

The equation (2.1.4) then follows. □

Let $\tilde{\vec{u}}$ be an approximation to $\vec{u}$ and $\tilde{\boldsymbol{\sigma}}$ to $\boldsymbol{\sigma}$. The Prager–Synge theorem states that if the conditions in the theorem are satisfied, then the two errors in the energy norm can be measured by the difference between the two approximations. The first condition $\tilde{\vec{u}} \in \boldsymbol{V}$ requires $\tilde{\vec{u}}$ to satisfy the conditions imposed in $\boldsymbol{V}$. To interpret the condition (2.1.3), first note that the equation

$$\int_{P^t} \tilde{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}(\vec{v}) = \int_{\Omega_+^t} q_+ v_3 + \int_{\Omega_-^t} q_- v_3$$

obviously holds for $\vec{v} \in \mathcal{R}$ (cf. (1.7)) as well as for $\vec{v} \in \boldsymbol{V}$. Therefore it holds for all smooth compactly supported functions on $P^t$, and so implies that

$$\overrightarrow{\mathrm{div}}\tilde{\boldsymbol{\sigma}} = \vec{0} \qquad \text{on } P^t. \tag{2.1.5}$$

Assuming that $\tilde{\boldsymbol{\sigma}} \in \boldsymbol{H}^1(P^t)$, integrating by parts in (2.1.3), this gives

$$\int_{P^t} \tilde{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}(\vec{v}) = \int_{\Omega_+^t} (\tilde{\boldsymbol{\sigma}}\vec{n}) \cdot \vec{v} + \int_{\Omega_-^t} (\tilde{\boldsymbol{\sigma}}\vec{n}) \cdot \vec{v} + \int_{\Gamma^t} (\tilde{\boldsymbol{\sigma}}\vec{n}) \cdot \vec{v} \qquad \text{for all } \vec{v} \in \boldsymbol{V}.$$

It follows that

$$\tilde{\boldsymbol{\sigma}}\vec{n} = (0, 0, q_\pm)^T \qquad \text{on } \Omega_\pm^t, \tag{2.1.6}$$

and

$$\int_{\Gamma^t} (\tilde{\boldsymbol{\sigma}} \vec{n}) \cdot \vec{v} = 0 \qquad \text{for all } \vec{v} \in \boldsymbol{V}. \tag{2.1.7}$$

Conversely, if $\tilde{\boldsymbol{\sigma}} \in \boldsymbol{H}^1(P^t)$ satisfies (2.1.5)–(2.1.7), then (2.1.3) holds.

The equation (2.1.7) implies that for each component of displacements not prescribed zero value the corresponding component of traction must be zero. In other words, $\tilde{\boldsymbol{\sigma}}$ must satisfy the boundary conditions on $\boldsymbol{\sigma}$. In particular, for the soft simply supported plate, this condition is specified in (1.5), for the hard simply supported plate, in (1.6).

## Sec 2.2. Expressions for $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\vec{u}}$

To apply the Prager–Synge theorem to the Kirchhoff plate convergence problems, $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\vec{u}}$ should be constructed from the Kirchhoff plate solution $w$. However, as we shall see later, explicit expressions constructed from $w$ may fail to satisfy the boundary conditions required by the theorem. Then boundary correctors are introduced to offset the undesired boundary values. Thus we will take

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{\mathrm{k}} + \boldsymbol{\sigma}^{\mathrm{c}}, \qquad \tilde{\vec{u}} = \vec{u}^{\mathrm{k}} + \vec{u}^{\mathrm{c}},$$

where $\boldsymbol{\sigma}^{\mathrm{k}}$ and $\vec{u}^{\mathrm{k}}$ are explicit approximations constructed from $w$, and $\boldsymbol{\sigma}^{\mathrm{c}}$ and $\vec{u}^{\mathrm{c}}$ are boundary correctors.

**Expression for $\boldsymbol{\sigma}^{\mathrm{k}}$.** Let $\boldsymbol{\sigma}$ be the three-dimensional stress field. The upper left $2 \times 2$ submatrix $(\sigma_{\alpha\beta})$ of $\boldsymbol{\sigma}$ represents in-plane stress. For the Kirchhoff plate the in-plane stress is expressed by

$$-\frac{Ez}{(1-\nu^2)} \left[ (1-\nu) \underset{\approx}{\mathrm{grad}} \, (\underset{\sim}{\mathrm{grad}} \, w) + \nu \Delta w \underset{\approx}{\delta} \right].$$

Thus let

$$\sigma^{\mathrm{k}}_{\alpha\beta} = -\frac{Ez}{(1-\nu^2)}\left[(1-\nu)\underset{\approx}{\mathrm{grad}}\,(\underset{\sim}{\mathrm{grad}}\,w) + \nu\Delta w\,\underset{\approx}{\delta}\right]. \tag{2.2.1}$$

We would like $\boldsymbol{\sigma}^{\mathrm{k}}$ to satisfy as many conditions of (2.1.5)–(2.1.7) as possible. One observes that $\vec{\mathrm{div}}\boldsymbol{\sigma}^{\mathrm{k}} = \vec{0}$ and $\boldsymbol{\sigma}^{\mathrm{k}}\vec{n} = (0,0,q_{\pm})^T$ on $\Omega^t_{\pm}$ can be obtained through determining $\sigma^{\mathrm{k}}_{\alpha 3}$ from $\sigma^{\mathrm{k}}_{\alpha\beta}$ and then determining $\sigma_{33}$ from $\sigma^{\mathrm{k}}_{\alpha 3}$. Thus $\boldsymbol{\sigma}^{\mathrm{k}}$ has the following expression:

$$\begin{aligned}
\sigma^{\mathrm{k}}_{11} &= -\frac{Ez}{1-\nu^2}\left[(1-\nu)\frac{\partial^2 w}{\partial x^2} + \nu\Delta w\right],\\
\sigma^{\mathrm{k}}_{22} &= -\frac{Ez}{1-\nu^2}\left[(1-\nu)\frac{\partial^2 w}{\partial y^2} + \nu\Delta w\right],\\
\sigma^{\mathrm{k}}_{33} &= \frac{zt^2 g}{2}\left(3 - \frac{4z^2}{t^2}\right),\\
\sigma^{\mathrm{k}}_{12} &= -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y},\\
\sigma^{\mathrm{k}}_{13} &= -\frac{E}{2(1-\nu^2)}\left(\frac{t^2}{4} - z^2\right)\frac{\partial \Delta w}{\partial x},\\
\sigma^{\mathrm{k}}_{23} &= -\frac{E}{2(1-\nu^2)}\left(\frac{t^2}{4} - z^2\right)\frac{\partial \Delta w}{\partial y}.
\end{aligned} \tag{2.2.2}$$

This $\boldsymbol{\sigma}^{\mathrm{k}}$ satisfies (2.1.5) and (2.1.6). However, it may not satisfy (2.1.7). Therefore later we shall define $\boldsymbol{\sigma}^{\mathrm{c}}$ such that

$$\begin{aligned}
&\vec{\mathrm{div}}\boldsymbol{\sigma}^{\mathrm{c}} = \vec{0} && \text{in } P^t,\\
&\boldsymbol{\sigma}^{\mathrm{c}}\vec{n} = \vec{0} && \text{on } \Omega^t_{+} \cup \Omega^t_{-},\\
&\int_{\Gamma^t} (\boldsymbol{\sigma}^{\mathrm{c}}\vec{n})\cdot\vec{v} = -\int_{\Gamma^t} (\boldsymbol{\sigma}^{\mathrm{k}}\vec{n})\cdot\vec{v} && \text{for all } \vec{v} \in \boldsymbol{V}.
\end{aligned} \tag{2.2.3}$$

Then $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{\mathrm{k}} + \boldsymbol{\sigma}^{\mathrm{c}}$ satisfies all the conditions of the Prager–Synge theorem on the stress field.

**Expression for $\vec{u}^{\mathrm{k}}$.** The construction for $\vec{u}^{\mathrm{k}}$ is made such that $\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E$ is of sufficiently high order.

Morgenstern [18], and Babuška and Pitkäranta [7] essentially used the same expressions for $\vec{u}^{\rm k}$. We shall use a slight modified version of their expression. We denote it by $\vec{u}^{\rm m}$.

$$\underset{\sim}{u}^{\rm m} = -z\,\underset{\sim}{\rm grad}\,w, \quad \vec{u}^{\rm m}\cdot\vec{e}_3 = w + \frac{\nu(12z^2-t^2)}{24(1-\nu)}\Delta w, \tag{2.2.4}$$

where $\underset{\sim}{u}^{\rm m}$ stands for $(u_1^{\rm m}, u_2^{\rm m})$.

From (2.2.4) and (2.2.2), a direct computation shows that there exists a constant $C$ independent of $t$ such that

$$\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\rm m}) - \boldsymbol{\sigma}^{\rm k}\|_E \le Ct^{5/2}\|g\|_{L^2(\Omega)}. \tag{2.2.5}$$

The last inequality follows from the standard regularity results for the biharmonic equation (1.9). See [7] for details. This $\vec{u}^{\rm m}$ may not satisfy the boundary conditions imposed in $\boldsymbol{V}$. Thus it is necessary to define $\vec{u}^{\rm c}$ so that $\tilde{\vec{u}} = \vec{u}^{\rm m} + \vec{u}^{\rm c} \in \boldsymbol{V}$ can be used in the Prager–Synge theorem.

Simmonds [21] studied a more sophisticated expression for $\vec{u}^{\rm k}$. We shall use a slightly modified version of his expression. Denote it by $\vec{u}^{\rm s}$.

$$\begin{aligned} \underset{\sim}{u}^{\rm s} &= -z\underset{\sim}{\rm grad}w + \left[\frac{zt^2}{1-\nu}\left(\frac{\nu}{24}-\frac{1}{4}\right) + \frac{z^3}{1-\nu}\left(\frac{1}{3}-\frac{\nu}{6}\right)\right]\underset{\sim}{\rm grad}\Delta w, \\ u_3^{\rm s} &= w + \frac{\nu(12z^2-t^2)}{24(1-\nu)}\Delta w, \end{aligned} \tag{2.2.6}$$

where $\underset{\sim}{u}^{\rm s}$ stands for $(u_1^{\rm s}, u_2^{\rm s})^T$.

**Lemma 2.2.1.** *Let $\boldsymbol{\sigma}^{\rm k}$ and $\vec{u}^{\rm s}$ be defined in* (2.2.2) *and* (2.2.6) *respectively, $w$ the Kirchhoff plate solution, $g$ the scaled traction in* (1.4). *Then there exists a constant $C$ independent of $t$ such that*

$$\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\rm s}) - \boldsymbol{\sigma}^{\rm k}\|_E \le Ct^{7/2}\|w\|_{H^4(\Omega)} \le Ct^{7/2}\|g\|_{L^2(\Omega)}. \tag{2.2.7}$$

*Proof.* Define $\vec{u}^{\mathrm{s}}$ by

$$\underset{\sim}{u}^{\mathrm{s}} = -z\underset{\sim}{\mathrm{grad}}\, w, \quad u_3^{\mathrm{s}} = u_3^{\mathrm{s}}. \tag{2.2.8}$$

Then a direct computation shows that

$$\left[A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{s}})\right]_{\alpha\lambda} = \sigma^{\mathrm{k}}_{\alpha\lambda}, \quad \left[A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{s}})\right]_{33} = 0. \tag{2.2.9}$$

Moreover,

$$\left[A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{s}})\right]_{\alpha 3} = \sigma^{\mathrm{k}}_{\alpha 3}. \tag{2.2.10}$$

By (2.2.8), (2.2.9), and (2.2.10) the inequality (2.2.7) follows from a simple computation and the triangle inequality. □

The expression for $\vec{u}^{\mathrm{m}}$ is simpler than that for $\vec{u}^{\mathrm{s}}$, and is good for some convergence rate estimation. However, when higher convergence rates are possible, the expression for $\vec{u}^{\mathrm{s}}$ is needed to obtain a higher order approximation like sharper results. Depending on the applications, one can choose which to use. For example, in the next section we shall use $\vec{u}^{\mathrm{s}}$ since the convergence rate for a bi-periodic plate is high, while in the next chapter, we shall use $\vec{u}^{\mathrm{m}}$ since the convergence rate finally turns out to be low and $\vec{u}^{\mathrm{m}}$ allows a simpler expression.

In addition to (2.2.5) or (2.2.7), for simply supported plates, since by (1.10) $w = 0$ on $\partial\Omega$, then

$$\int_{\Gamma^t} \vec{u}^{\mathrm{m}} = \vec{u}^{\mathrm{s}} = (0, 0, tw)^T = \vec{0}. \tag{2.2.11}$$

While the estimation (2.2.5) and (2.2.7) holds for Morgenstern's expression and Simmonds's expression respectively, only our modified versions satisfy both (2.2.5) and (2.2.11) or both (2.2.7) and (2.2.11). The equation (2.2.11) will be an essential condition in our three-dimensional boundary value corrector discussion. Once again $\vec{u}^{\mathrm{k}}$ may not satisfy the required boundary condition, and we require a boundary corrector $\vec{u}^{\mathrm{c}}$.

## Sec 2.3. Convergence Rate for a Bi-periodic Plate

In this section, we consider a bi-periodic plate. When the expressions $\sigma^{\mathrm{k}}$ and $\vec{u}^{\mathrm{s}}$ are employed, the Prager–Synge theorem leads to the global convergence rate of $O(t^2)$.

For the bi-periodic plate, let $\Omega$ be the unit square $(0,1)\times(0,1)$. Define $\boldsymbol{\Sigma}$ by (2.1.1), and

$$\boldsymbol{V}=\Big\{\vec{v}\,|\,v_i\in H^1(P^t), \vec{v}(0,y,z)=\vec{v}(1,y,z),\ \text{ for all } 0<y<1, -\frac{t}{2}<z<\frac{t}{2},$$
$$\vec{v}(x,0,z)=\vec{v}(x,1,z),\ \text{ for all } 0<x<1, -\frac{t}{2}<z<\frac{t}{2} \text{ and } \int_{P^t}\vec{v}=\vec{0}\Big\}. \tag{2.3.1}$$

Suppose that $g$ satisfies

$$g(0,y)=g(1,y), \qquad \text{for all } 0<y<1,$$
$$g(x,0)=g(x,1), \qquad \text{for all } 0<x<1,$$
$$\int_\Omega g=0.$$

Then the weak formulation (2.1.2) determines the solution $(\boldsymbol{\sigma},\vec{u})$ uniquely.

The requirement (2.1.7) corresponds to the following bi-periodic condition:

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\vec{n})(0,y,z)&=-(\boldsymbol{\sigma}\cdot\vec{n})(1,y,z) \qquad \text{for all } 0<y<1,\ -\frac{t}{2}<z<\frac{t}{2},\\ (\boldsymbol{\sigma}\cdot\vec{n})(x,0,z)&=-(\boldsymbol{\sigma}\cdot\vec{n})(x,1,z) \qquad \text{for all } 0<x<1,\ -\frac{t}{2}<z<\frac{t}{2}.\end{aligned} \tag{2.3.2}$$

We now check that in this special case $\boldsymbol{\sigma}^{\mathrm{k}}$ satisfies the conditions in (2.3.2) and $\vec{u}^{\mathrm{s}}$ satisfies the conditions in (2.3.1).

The corresponding Kirchhoff plate finds the unique solution $w\in\hat{H}^2_{\mathrm{per}}(\Omega)$ to the equation (1.9) where

$$\hat{H}^2_{\mathrm{per}}(\Omega)=\Big\{v\in H^2(\Omega)\,|\,v(0,y)=v(1,y),\ \frac{\partial v}{\partial x}(0,y)=\frac{\partial v}{\partial x}(1,y),\ 0<y<1,$$
$$v(x,0)=v(x,1),\ \frac{\partial v}{\partial x}(x,0)=\frac{\partial v}{\partial x}(x,1),\ 0<x<1,\ \int_\Omega v=0\Big\}.$$

By using the weak formulation for (1.9) and integrating by parts, it is easy to check that

$$
\begin{aligned}
(\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\operatorname{grad}}\, w)(0,y) &= (\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\operatorname{grad}}\, w)(1,y) && \text{for all } 0<y<1,\\
(\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\operatorname{grad}}\, w)(x,0) &= (\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\operatorname{grad}}\, w)(x,1) && \text{for all } 0<x<1,\\
(\underset{\sim}{\operatorname{grad}}\, \Delta w)(0,y) &= (\underset{\sim}{\operatorname{grad}}\, \Delta w)(1,y) && \text{for all } 0<y<1,\\
(\underset{\sim}{\operatorname{grad}}\, \Delta w)(x,0) &= (\underset{\sim}{\operatorname{grad}}\, \Delta w)(x,1) && \text{for all } 0<x<1.
\end{aligned}
$$

By (2.2.1), (2.2.2) and (2.2.6), the components of $\boldsymbol{\sigma}^{\mathrm{k}}$ and $\vec{u}^{\mathrm{s}}$ are all linear combinations of $w$, $\dfrac{\partial w}{\partial x}$, $\dfrac{\partial w}{\partial y}$, $\Delta w$, $\underset{\sim}{\operatorname{grad}}\Delta w$. Thus,

$$
\begin{aligned}
(\boldsymbol{\sigma}^{\mathrm{k}}\vec{n})(0,y,z) &= -(\boldsymbol{\sigma}^{\mathrm{k}}\vec{n})(1,y,z) && \text{for all } 0<y<1,\ -\frac{t}{2}<z<\frac{t}{2},\\
(\boldsymbol{\sigma}^{\mathrm{k}}\vec{n})(x,0,z) &= -(\boldsymbol{\sigma}^{\mathrm{k}}\vec{n})(x,1,z) && \text{for all } 0<x<1,\ -\frac{t}{2}<z<\frac{t}{2},\\
\vec{u}^{\mathrm{s}}(0,y,z) &= \vec{u}^{\mathrm{s}}(1,y,z), && \text{for all } 0<y<1,\ -\frac{t}{2}<z<\frac{t}{2},\\
\vec{u}^{\mathrm{s}}(x,0,z) &= \vec{u}^{\mathrm{s}}(x,1,z) && \text{for all } 0<x<1,\ -\frac{t}{2}<z<\frac{t}{2}.
\end{aligned}
$$

Moreover, by (2.2.6) and the fact that $\int_\Omega w = 0$, it is easy to check that

$$\int_{P^t} \vec{u}^{\mathrm{s}} = \vec{0}.$$

Thus, the conditions in (2.3.2) and (2.3.1) are all satisfied. Hence $\boldsymbol{\sigma}^{\mathrm{k}}$ and $\vec{u}^{\mathrm{s}}$ can be used directly in the Prager–Synge theorem with boundary corrector $\boldsymbol{\sigma}^{\mathrm{c}} = \mathbf{0}$ and $\vec{u}^{\mathrm{s}} = \vec{0}$. By (2.2.7), we obtain the following result for the bi-periodic plate.

**Theorem 2.3.1.** *Let $(\boldsymbol{\sigma}, \vec{u})$ be the solution to* (2.1.2)*, $(\boldsymbol{\sigma}^{\mathrm{k}}, \vec{u}^{\mathrm{s}})$ be constructed in* (2.2.2) *and* (2.2.6) *from the Kirchhoff plate solution. Then there exists a constant $C$ independent of $t$ such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u} - \vec{u}^{\mathrm{s}}||| \le C t^{7/2} \|g\|_{L^2(\Omega)}.$$

□

**Corollary 2.3.2.** *Let* $(\boldsymbol{\sigma}, \vec{u})$ *be same as in Theorem 3.2.1. Then there exists a constant* $C$ *independent of* $t$ *such that*

$$\|\boldsymbol{\sigma}\|_E + |||\vec{u}||| \le Ct^{3/2}\|g\|_{L^2(\Omega)}.$$

*Proof.* From (2.2.2) and (2.2.6), it is easy to see that

$$\|\boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u}^{\mathrm{s}}||| \le Ct^{3/2}\|g\|_{L^2(\Omega)}.$$

The corollary then follows from Theorem 2.2.1 immediately. □

From Theorem 2.2.1 and Corollary 2.2.2, the convergence rate follows.

**Theorem 2.3.3.** *Under the conditions of Theorem 2.2.1. The global convergence rate for the periodic plate is* $O(t^2)$*. That is, there exists a constant* $C$ *independent of* $t$ *such that*

$$\frac{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E}{\|\boldsymbol{\sigma}\|_E} + \frac{|||\vec{u} - \vec{u}^{\mathrm{s}}|||}{|||\vec{u}|||} \le Ct^2.$$

□

Note that if we replace $\vec{u}^{\mathrm{s}}$ by $\vec{u}^{\mathrm{m}}$ in this section, all the analysis holds, except that the convergence rate we can obtain is of order $O(t)$, which is not sharp.

## Sec 2.4. Boundary Correctors

The easy application of the Prager–Synge theorem and high convergence rate of the bi-periodic plate in section 2.3 is a rare exception. For most boundary value problems, possible boundary layers of the three-dimensional solution make it impossible to use $\boldsymbol{\sigma}^{\mathrm{k}}$ and $\vec{u}^{\mathrm{m}}$ (or $\vec{u}^{\mathrm{s}}$) directly in the Prager–Synge theorem. Boundary correctors are usually necessary.

**Boundary values to be corrected for the simply supported plates.** We compare the boundary conditions on the three-dimensional solution $(\boldsymbol{\sigma}, \vec{u})$ with the corresponding values of $(\boldsymbol{\sigma}^{\mathrm{k}}, \vec{u}^{\mathrm{m}})$, or $(\boldsymbol{\sigma}^{\mathrm{k}}, \vec{u}^{\mathrm{s}})$. Then we find the boundary values that need to be corrected.

By (1.5) and (1.6), the boundary values imposed on the lateral side $\Gamma^t$ are

$$\begin{aligned} \vec{u} \cdot \vec{e}_3 = \vec{s}^T \boldsymbol{\sigma} \vec{n} = \vec{n}^T \boldsymbol{\sigma} \vec{n} = 0 \qquad & \text{soft case,} \\ \vec{u} \cdot \vec{e}_3 = \vec{u} \cdot \vec{s} = \vec{n}^T \boldsymbol{\sigma} \vec{n} = 0 \qquad & \text{hard case.} \end{aligned}$$

By (1.10), the simply supported Kirchhoff plate solution satisfies

$$w = 0, \qquad \underset{\sim}{n}^T \left[ (1-\nu) \underset{\approx}{\mathrm{grad}} (\underset{\sim}{\mathrm{grad}} w) + \nu \Delta w \underset{\approx}{\delta} \right] \underset{\sim}{n} = 0 \qquad \text{on } \partial\Omega.$$

Thus (2.2.1), on the lateral side $\Gamma^t$,

$$\begin{aligned} & \vec{s}^T \boldsymbol{\sigma}^{\mathrm{k}} \vec{n} = z \underset{\sim}{s}^T (T \underset{\approx}{\mathrm{grad}} \underset{\sim}{\mathrm{grad}} w) \underset{\sim}{n}, \\ & \vec{n}^T \boldsymbol{\sigma}^{\mathrm{k}} \vec{n} = 0, \end{aligned} \tag{2.4.1}$$

where

$$T \underset{\approx}{\tau} = \frac{E}{(1-\nu^2)} \left[ (1-\nu) \underset{\approx}{\tau} + \nu \operatorname{tr} \underset{\approx}{\tau} \underset{\approx}{\delta} \right].$$

By (2.2.4), on the lateral side $\Gamma^t$,

$$\begin{aligned} & \vec{u}^{\mathrm{m}} \cdot \vec{e}_3 = \frac{\nu(12z^2 - t^2)}{24(1-\nu)} \Delta w, \\ & \vec{u}^{\mathrm{m}} \cdot \vec{s} = 0, \end{aligned} \tag{2.4.2}$$

where $\vec{s} = (s_1, s_2, 0)$, $\underset{\sim}{s} = (s_1, s_2)$, $\vec{n} = (n_1, n_2, 0)$, $\underset{\sim}{n} = (n_1, n_2)$. By (2.2.6), on the lateral side $\Gamma^t$,

$$\begin{aligned} & \vec{u}^{\mathrm{s}} \cdot \vec{e}_3 = \frac{\nu(12z^2 - t^2)}{24(1-\nu)} \Delta w, \\ & \vec{u}^{\mathrm{s}} \cdot \vec{s} = \left[ \frac{zt^2}{1-\nu} \left( \frac{\nu}{24} - \frac{1}{4} \right) + \frac{z^3}{1-\nu} \left( \frac{1}{3} - \frac{\nu}{6} \right) \right] \underset{\sim}{\mathrm{grad}} \Delta w \cdot \underset{\sim}{s}. \end{aligned} \tag{2.4.3}$$

The nonzero values on the right hand side of (2.4.1) and (2.4.2) or (2.4.1) and (2.4.3) are the values to be corrected by the boundary correctors $\boldsymbol{\sigma}^{\mathrm{c}}$ and $\vec{u}^{\mathrm{c}}$.

From now on we shall use $\vec{u}^{\mathrm{k}}$ for either $\vec{u}^{\mathrm{m}}$ or $\vec{u}^{\mathrm{k}}$. All the discussion applies to both of them. Define the following functions on $\Gamma^t$:

$$\frac{1}{t} h_3 = -\vec{u}^{\mathrm{k}} \cdot \vec{e}_3, \quad h_s = -\vec{u}^{\mathrm{k}} \cdot \vec{s}, \quad f = -\vec{s}^T \boldsymbol{\sigma}^{\mathrm{k}} \vec{n}. \tag{2.4.4}$$

We will see the reason for scaling $h_3$ in section 3.2. Since the variable $z$, whose range is $[-t/2, t/2]$, is an order $O(t)$ term, $h_3$ is of order $O(t^3)$ pointwise, $h_s$ is either zero or order $O(t^3)$ pointwise, and $f$ is of order $O(t)$ pointwise.

**Applying the Prager–Synge theorem.** We will take $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{\mathrm{k}} + \boldsymbol{\sigma}^{\mathrm{c}}$ and $\tilde{\vec{u}} = \vec{u}^{\mathrm{k}} + \vec{u}^{\mathrm{c}}$ in the Prager–Synge theorem. By the theorem,

$$\begin{aligned} &|||\vec{u} - (\vec{u}^{\mathrm{k}} + \vec{u}^{\mathrm{c}})|||^2 + \|\boldsymbol{\sigma} - (\boldsymbol{\sigma}^{\mathrm{k}} + \boldsymbol{\sigma}^{\mathrm{c}})\|_E^2 = \|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}} + \vec{u}^{\mathrm{c}}) - (\boldsymbol{\sigma}^{\mathrm{k}} + \boldsymbol{\sigma}^{\mathrm{c}})\|_E^2 \\ &\quad \le 2\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E^2 + 2\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E^2. \end{aligned}$$

The term $\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E^2$ is bounded by either (2.2.5) or (2.2.7). Thus by the triangle inequality,

$$\begin{aligned} |||\vec{u} - \vec{u}^{\mathrm{k}}||| &\le \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E + \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u}^{\mathrm{c}}||| \\ &\le \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E + \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u}^{\mathrm{c}}|||, \end{aligned} \tag{2.4.5}$$

and

$$\begin{aligned} |||\vec{u} - \vec{u}^{\mathrm{k}}||| &\ge -\sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E - \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u}^{\mathrm{c}}||| \\ &\ge -\sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E - \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u}^{\mathrm{c}}|||. \end{aligned} \tag{2.4.6}$$

Likewise,

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E \le \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E + \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + \|\boldsymbol{\sigma}^{\mathrm{c}}\|_E, \tag{2.4.7}$$

and

$$|||\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}||| \geq -\sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E - \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + \|\boldsymbol{\sigma}^{\mathrm{c}}\|_E. \quad (2.4.8)$$

A similar approach can be used to obtain energy estimates on a subdomain $P_0^t$. Since

$$|||\vec{u} - \vec{u}^{\mathrm{k}}|||_{P_0^t} \leq |||\vec{u} - \vec{u}^{\mathrm{k}} - \vec{u}^{\mathrm{c}}|||_{P_0^t} + |||\vec{u}^{\mathrm{c}}|||_{P_0^t} \leq |||\vec{u} - \vec{u}^{\mathrm{k}} - \vec{u}^{\mathrm{c}}||| + |||\vec{u}^{\mathrm{c}}|||_{P_0^t},$$

then

$$|||\vec{u} - \vec{u}^{\mathrm{k}}|||_{P_0^t} \leq \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E + \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u}^{\mathrm{c}}|||_{P_0^t}, \quad (2.4.9)$$

$$|||\vec{u} - \vec{u}^{\mathrm{k}}|||_{P_0^t} \geq -\sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E - \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u}^{\mathrm{c}}|||_{P_0^t}. \quad (2.4.10)$$

Or,

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_{E,P_0^t} \leq \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E + \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + \|\boldsymbol{\sigma}^{\mathrm{c}}\|_{E,P_0^t}, \quad (2.4.11)$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_{E,P_0^t} \geq -\sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E - \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + \|\boldsymbol{\sigma}^{\mathrm{c}}\|_{E,P_0^t}. \quad (2.4.12)$$

In either [7] or [18], $\boldsymbol{\sigma}^{\mathrm{c}}$ and $\vec{u}^{\mathrm{c}}$ are considered separately. $\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E$ and $|||\vec{u}^{\mathrm{c}}|||$ are found to be of the same order $O(t^2)$. Although this leads to a global convergence rate of $O(t^{1/2})$, this approach does not establish the sharpness of the convergence rate. Moreover, it cannot be used to obtain higher interior convergence rates.

We notice the simple fact that it is desirable to find $\boldsymbol{\sigma}^{\mathrm{c}}$ and $\vec{u}^{\mathrm{c}}$ such that the order of $\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E$ is higher than $|||\vec{u}^{\mathrm{c}}|||$ or $\|\boldsymbol{\sigma}^{\mathrm{c}}\|_E$ for global estimation, and higher than $|||\vec{u}^{\mathrm{c}}|||_{P_0^t}$ or $\|\boldsymbol{\sigma}^{\mathrm{c}}\|_{E,P_0^t}$ for interior estimation, so that the orders of $\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E$ and $\|\boldsymbol{\sigma}^{\mathrm{c}}\|_E$ or orders of $\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{k}}) - \boldsymbol{\sigma}^{\mathrm{k}}\|_E$ and $\|\boldsymbol{\sigma}^{\mathrm{c}}\|_{E,P_0^t}\|_E$ can be compared. This leads us to seek the following:

$$A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}} = \mathbf{0} \quad \text{in } P^t. \quad (2.4.13)$$

Recall that $\boldsymbol{\sigma}^{\mathrm{c}}$ must satisfy the conditions in (2.2.3). Then the equations in (2.4.13) and (2.2.3) determine $(\boldsymbol{\sigma}^{\mathrm{c}}, \vec{u}^{\mathrm{c}})$ as the solution to a three-dimensional plate problem. This problem consists of the differential equation

$$\vec{\operatorname{div}}\boldsymbol{\sigma}^{\mathrm{c}} = \vec{0} \quad \text{in } P^t, \tag{2.4.14}$$

the boundary condition on the top and bottom surfaces

$$\boldsymbol{\sigma}^{\mathrm{c}}\vec{n} = \vec{0} \quad \text{on } \Omega^t_+ \cup \Omega^t_-, \tag{2.4.15}$$

and on the lateral boundary $\Gamma^t$, the boundary values to be corrected are imposed. For the soft simply supported plate, they are

$$\begin{aligned} \vec{u}^{\mathrm{c}} \cdot \vec{e}_3 &= \frac{1}{t}h_3 \qquad &&\text{on } \Gamma^t, \\ \vec{s}^T\boldsymbol{\sigma}^{\mathrm{c}}\vec{n} &= f &&\text{on } \Gamma^t, \\ \vec{n}^T\boldsymbol{\sigma}^{\mathrm{c}}\vec{n} &= 0 &&\text{on } \Gamma^t. \end{aligned} \tag{2.4.16}$$

For the hard simply supported plate, they are

$$\begin{aligned} \vec{u}^{\mathrm{c}} \cdot \vec{e}_3 &= \frac{1}{t}h_3 \quad &&\text{on } \Gamma^t, \\ \vec{u}^{\mathrm{c}} \cdot \vec{s} &= h_s &&\text{on } \Gamma^t, \\ \vec{n}^T\boldsymbol{\sigma}^{\mathrm{c}}\vec{n} &= 0 &&\text{on } \Gamma^t. \end{aligned} \tag{2.4.17}$$

These boundary values are specified in(2.4.2) or (2.4.3). Finally, for the soft simply supported plate, we need to impose the side condition

$$\int_{P^t} \vec{u}^{\mathrm{c}} \cdot \vec{r} = -\int_{P^t} \vec{u}^{\mathrm{k}} \cdot \vec{r} \qquad \text{for all } \vec{r} \in \mathcal{R}.$$

Since $\underset{\sim}{u}^{\mathrm{k}}$ is odd in $z$, by (1.7), the right hand side of this condition becomes

$$\int_{P^t} \vec{u}^{\mathrm{c}} \cdot \vec{r} = 0 \qquad \text{for all } \vec{r} \in \mathcal{R}. \tag{2.4.18}$$

The equations (2.4.13)–(2.4.15), (2.4.16), and (2.4.18) uniquely determine the boundary corrector $(\boldsymbol{\sigma}^{\mathrm{c}}, \vec{u}^{\mathrm{c}})$ for the soft simply supported plate, and the equations (2.4.13)–(2.4.15), and (2.4.17) uniquely determines the boundary corrector $(\boldsymbol{\sigma}^{\mathrm{c}}, \vec{u}^{\mathrm{c}})$ for the hard simply supported plate.

The following theorem explains how the three-dimensional boundary corrector affects the error estimation.

**Theorem 2.4.1.** *Let* $(\boldsymbol{\sigma}, \vec{u})$ *be the solution to* (1.1)–(1.4), (1.5) *or to* (1.1)–(1.4), (1.6), $\boldsymbol{\sigma}^{\mathrm{k}}$ *as in* (2.2.2), $\vec{u}^{\mathrm{k}}$ *as in* (2.2.6), $(\boldsymbol{\sigma}^{\mathrm{c}}, \vec{u}^{\mathrm{c}})$ *as in* (2.4.13)–(2.4.15), (2.4.16),, *and* (2.4.18), *or as in* (2.4.13)–(2.4.15), *and* (2.4.17). *Then there exists a constant* $C$ *independent of* $t$ *such that*

$$2\sqrt{2}\|\boldsymbol{\sigma}^{\mathrm{c}}\|_E - \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E \leq |||\vec{u} - \vec{u}^{\mathrm{k}}||| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E$$
$$\leq 2\sqrt{2}\|\boldsymbol{\sigma}^{\mathrm{c}}\|_E + \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E.$$

*Moreover, let* $P_0^t$ *be an interior domain of the plate,*

$$\begin{aligned} 2\sqrt{2}\|\boldsymbol{\sigma}^{\mathrm{c}}\|_{E,P_0^t} - \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E &\leq |||\vec{u} - \vec{u}^{\mathrm{k}}|||_{P_0^t} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_{E,P_0^t} \\ &\leq 2\sqrt{2}\|\boldsymbol{\sigma}^{\mathrm{c}}\|_{E,P_0^t} + \sqrt{2}\|A^{-1}\boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) - \boldsymbol{\sigma}^{\mathrm{c}}\|_E. \end{aligned} \tag{2.4.19}$$

*Proof.* The inequality for the global estimation follows from (2.4.5)–(2.4.8) and (2.4.13). The inequality for the local estimation follows from (2.4.9)–(2.4.12) and (2.4.13). □

Note that the right hand side of (2.4.19) are to be used together with (2.2.5) or (2.2.7), depending on whether $\vec{u}^{\mathrm{k}} = \vec{u}^{\mathrm{m}}$ or $\vec{u}^{\mathrm{k}} = \vec{u}^{\mathrm{s}}$. As we shall see in chapter 3, the order of $\|\boldsymbol{\sigma}^{\mathrm{c}}\|_E$ and $\|\boldsymbol{\sigma}^{\mathrm{c}}\|_{E,P_0^t}$ do not exceed $O(t^{5/2})$. Thus the global and interior convergence rates are determined by the order of $\|\boldsymbol{\sigma}^{\mathrm{c}}\|_E$ and $\|\boldsymbol{\sigma}^{\mathrm{c}}\|_{E,P_0^t}$ respectively. The orders of these terms are not easy to estimate. It is the main work in the next chapter.

# Chapter Three

## CONVERGENCE RATES FOR SIMPLY SUPPORTED PLATES

In this chapter, we estimate the asymptotic orders of the boundary value correctors, and derive the convergence rates for the simply supported plates. It is found that for the hard simply supported plate with smooth boundary, the global convergence rate of $O(t^{1/2})$ is sharp, and the interior convergence rate is $O(t)$, while for the soft simply supported plate both the global and interior convergence rates are $O(t^{1/2})$.

The analysis in this chapter uses asymptotic methods employed by Destuynder and Ciarlet [11], [10]. Destuynder discussed convergence rates for the hard clamped plate, while we treat the cases of soft and hard simply supported plates. However, instead of correcting the boundary values arising from the second asymptotic expansion term in Destuynder's analysis, we correct the boundary values arising from the application of Prager–Synge theorem. Initiated by the work of Toupin [22] and Wan [16], [15], [14], our analysis also use the Saint Venant's principle, which makes the analysis clearer.

Following the approaches in [11], we organize this chapter as follows. In section 3.1, we recall the three-dimensional boundary corrector and its related properties. In section 3.2, a set of scalings is used to fix the plate thickness, and assign appropriate orders to the components of the displacement and the stress. In section 3.3, an auxiliary problem on a neighborhood of the lateral boundary of the plate is defined. The solution to this problem will agree with the scaled boundary corrector to the lowest order as that of $\boldsymbol{\sigma}^t$. To facilitate the discussion, a boundary-fitted coordinate system is used. In section 3.4, the auxiliary problem is decoupled into one two-dimensional Laplace-like problem and one two-dimensional elasticity-like problem with variable $\theta$ as a parameter. In section 3.5, 3.6, and 3.7, we analyze the two problems using Saint Venant's principle and find the orders of the solutions in various norms. In section 3.8, we estimate the difference between the solution to the auxiliary problem and $\boldsymbol{\sigma}^t$. In section 3.9, we obtain the order estimations about $\boldsymbol{\sigma}^{\mathrm{c}}$ from the results in the section 3.8. Finally in Theorem 2.4.1, the convergence rate results follows.

## Sec 3.1. Three-dimensional Boundary Correctors

As discussed in Section 2.4, the boundary corrector $(\boldsymbol{\sigma}^{\mathrm{c}}, \vec{u}^{\mathrm{c}})$ is the solution of a three-dimensional elasticity problem. The differential equations and plate domain are the same as the original three-dimensional plate problem, but the boundary conditions are different. There is no surface loading on the top and bottom surfaces $\Omega^t_+$ and $\Omega^t_-$, but nonzero data is given on the lateral boundary $\Gamma^t$. Specifically $(\boldsymbol{\sigma}^{\mathrm{c}}, \vec{u}^{\mathrm{c}})$ solves

$$\begin{aligned} A\boldsymbol{\sigma}^{\mathrm{c}} &= \boldsymbol{\varepsilon}(\vec{u}^{\mathrm{c}}) && \text{in } P^t, \\ \vec{\operatorname{div}}\boldsymbol{\sigma}^{\mathrm{c}} &= \mathbf{0} && \text{in } P^t \\ \boldsymbol{\sigma}^{\mathrm{c}}\vec{n} &= 0 && \text{on } \Omega^t_+ \cup \Omega^t_-. \end{aligned} \tag{3.1.1}$$

In view of (2.4.16), (2.4.17), (2.4.1), and (2.4.2) or (2.4.3), the boundary conditions imposed on $\Gamma^t$ can be written as follows:

$$\begin{aligned} &\vec{n}^T\boldsymbol{\sigma}^{\mathrm{c}}\vec{n} = 0, \quad \vec{s}^T\boldsymbol{\sigma}^{\mathrm{c}}\vec{n} = f, \quad \vec{u}^{\mathrm{c}}\cdot\vec{e}_3 = \frac{1}{t}h_3 \quad \text{soft case,} \\ &\vec{n}^T\boldsymbol{\sigma}^{\mathrm{c}}\vec{n} = 0, \quad \vec{u}^{\mathrm{c}}\cdot\vec{s} = h_s, \quad \vec{u}^{\mathrm{c}}\cdot\vec{e}_3 = \frac{1}{t}h_3 \quad \text{hard case,} \end{aligned} \tag{3.1.2}$$

where $h_s$ and $h_3$ and $f$ are defined in (2.4.4). Note the following facts which follow from Section 2.4.

(1) All three functions $h_s$, $h_3$, and $f$ are determined by the first, second, and third order derivatives of the Kirchhoff plate solution $w$.

(2) The functions $h_s$ and $f$ are odd functions of $z$ while $h_3$ is an even function of $z$.

(3) The function $h_3$ is of order $O(t^3)$ pointwise, $h_s$ is either zero or of order $O(t^3)$ pointwise, and $f$ is of order $O(t)$ pointwise.

(4) The integrals of $h_s$, $h_3$, and $f$ with respect to $z$ ranging from $-t/2$ to $t/2$ vanish identically.

The weak formulation for (3.1.1) and (3.1.2) is the following:

Find $(\boldsymbol{\sigma}^c, \vec{u}^c)$ such that $\vec{u}^c$ satisfies the boundary conditions in (3.1.2) and

$$\begin{aligned} &\int_{P^t} A\boldsymbol{\sigma}^c : \boldsymbol{\tau} - \int_{P^t} \boldsymbol{\varepsilon}(\vec{u}^c) : \boldsymbol{\tau} = 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \\ &\int_{P^t} \boldsymbol{\sigma}^c : \boldsymbol{\varepsilon}(\vec{v}) = \begin{cases} \int_\Gamma f(\vec{v}\cdot\vec{s}) & \text{soft case} \\ 0 & \text{hard case} \end{cases} && \text{for all } \vec{v} \in \boldsymbol{V}. \end{aligned} \tag{3.1.3}$$

where the displacement variable space $\boldsymbol{V}$ and stress field space $\boldsymbol{\Sigma}$ are given in (2.1.1).

For (3.1.3) to be equivalent to (3.1.1) and (3.1.2), it is necessary for (3.1.3) to hold for $\vec{v} \in \mathcal{R}$. That is

$$\int_{\Gamma^t} f(\vec{r}\cdot\vec{s}) = 0 \qquad \text{for all } \vec{r} \in \mathcal{R}. \tag{3.1.4}$$

Since $\vec{r}\cdot\vec{s}$ is independent of $z$ for any $\vec{r} \in \mathcal{R}$, (3.1.4) holds.

## Sec 3.2. Scaling

Following Ciarlet [10] and Destuynder [11], we scale the dependent and independent variables. The coordinate variables $x$, $y$, $z$ are scaled to $x_1$, $x_2$ and $x_3$ as follows:

$$x_1 = x, \quad x_2 = y, \quad x_3 = \frac{z}{t}. \tag{3.2.1}$$

Thus the plate domain $P^t$ is scaled to $P = \Omega \times (-1/2, 1/2)$. Denote the top surface of $P$ by $\Omega_+$ and bottom surface of $P$ by $\Omega_-$.

The displacement and the stress are scaled as follows:

$$\begin{aligned}
&\sigma^t_{\alpha\beta}(x_1,x_2,x_3) = \sigma^c_{\alpha\beta}(x_1,x_2,tx_3), \;\; \sigma^t_{\alpha 3}(x_1,x_2,x_3) = \frac{1}{t}\sigma^c_{\alpha 3}(x_1,x_2,tx_3),\\
&\sigma^t_{33}(x_1,x_2,x_3) = \frac{1}{t^2}\sigma^c_{33}(x_1,x_2,tx_3), \qquad u^t_\alpha(x_1,x_2,x_3) = u^c_\alpha(x_1,x_2,tx_3),\\
&u^t_3(x_1,x_2,x_3) = tu^c_3(x_1,x_2,tx_3).
\end{aligned} \tag{3.2.2}$$

The boundary data becomes

$$\begin{aligned}
&h^t_s(x_1,x_2,x_3) = h_s(x_1,x_2,tx_3), \quad h^t_3(x_1,x_2,x_3) = h_3(x_1,x_2,tx_3),\\
&f^t(x_1,x_2,x_3) = f(x_1,x_2,tx_3).
\end{aligned} \tag{3.2.3}$$

With this scaling, $\vec{u}^t \cdot \vec{e}_3$ assumes the value $h^t_3$ on $\Gamma$. This is the reason for the factor of $1/t$ in (3.1.2).

With such scalings, the spaces $\boldsymbol{V}$, $\boldsymbol{\Sigma}$ become $\boldsymbol{V}^t$, $\boldsymbol{\Sigma}^t$:

$$\begin{aligned}
&\boldsymbol{\Sigma}^t = \left\{ \boldsymbol{\tau} \,|\, \tau_{ij} \in L^2(P), \tau_{ij} = \tau_{ji} \right\},\\
&\boldsymbol{V}^t = \left\{ \vec{v} \,|\, v_i \in H^1(P), \vec{v}\cdot\vec{e}_3 = 0 \text{ on } \Gamma, \text{ (1.8) holds} \right\} \quad \text{soft case,}\\
&\boldsymbol{V}^t = \left\{ \vec{v} \,|\, v_i \in H^1(P), \vec{v}\cdot\vec{e}_3 = \vec{v}\cdot\vec{s} = 0 \text{ on } \Gamma \right\} \quad \text{hard case.}
\end{aligned} \tag{3.2.4}$$

The scaled corrector $(\boldsymbol{\sigma}^t, \vec{u}^t)$ is the solution to the following problem:

$$\begin{aligned}
\vec{\text{div}}\boldsymbol{\sigma}^t &= \vec{0} && \text{in } P,\\
\boldsymbol{\sigma}^t\vec{n} &= \vec{0} && \text{on } \Omega_+ \cup \Omega_-.
\end{aligned} \tag{3.2.5}$$

with the following lateral boundary conditions:

$$\begin{aligned}
&\vec{n}^T\boldsymbol{\sigma}^t\vec{n} = 0, \quad \vec{s}^T\boldsymbol{\sigma}^t\vec{n} = f^t, \quad \vec{u}^t\cdot\vec{e}_3 = h^t_3 \quad \text{soft case,}\\
&\vec{n}^T\boldsymbol{\sigma}^t\vec{n} = 0, \quad \vec{u}^t\cdot\vec{s} = h^t_s, \quad \vec{u}^t\cdot\vec{e}_3 = h^t_3 \quad \text{hard case,}
\end{aligned} \tag{3.2.6}$$

where the components of $\boldsymbol{\sigma}^t$ in (3.2.5) and (3.2.6) are as follows:

$$\begin{aligned}
\sigma^t_{\alpha\beta} &= \frac{E}{2(1+\nu)}\left[\frac{\partial u^t_\beta}{\partial x_\alpha} + \frac{\partial u^t_\alpha}{\partial x_\beta} + \frac{2\nu}{1-2\nu}\left(\frac{\partial u^t_\lambda}{\partial x_\lambda} + \frac{1}{t^2}\frac{\partial u^t_3}{\partial x_3}\right)\delta_{\alpha\beta}\right],\\
\sigma^t_{\alpha 3} &= \frac{E}{2(1+\nu)t^2}\left(\frac{\partial u^t_3}{\partial x_\alpha} + \frac{\partial u^t_\alpha}{\partial x_3}\right),\\
\sigma^t_{33} &= \frac{E}{(1+\nu)t^4}\left[\frac{\partial u^t_3}{\partial x_3} + \frac{\nu}{1-2\nu}\left(\frac{\partial u^t_3}{\partial x_3} + t^2\frac{\partial u^t_\alpha}{\partial x_\alpha}\right)\right].
\end{aligned} \tag{3.2.7}$$

**Lemma 3.2.1.** *The problem* (3.2.5)–(3.2.7) *has the weak formulation:*

*Find* $(\boldsymbol{\sigma}^t, \vec{u^t})$ *such that* $\vec{u^t}$ *satisfies the boundary conditions in* (3.2.6), *and* $(\boldsymbol{\sigma}^t, \vec{u^t})$ *satisfies the following equations:*

$$\begin{aligned} &a_0(\boldsymbol{\sigma}^t, \boldsymbol{\tau}) + t^2 a_2(\boldsymbol{\sigma}^t, \boldsymbol{\tau}) + t^4 a_4(\boldsymbol{\sigma}^t, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \vec{u^t}) = 0, \quad \textit{for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma}^t, \\ &b(\boldsymbol{\sigma}^t, \vec{v}) = \begin{cases} \int_\Gamma f^t(\vec{v} \cdot \vec{s}) & \textit{soft case} \\ 0 & \textit{hard case} \end{cases} \quad \textit{for all } \vec{v} \in \boldsymbol{V}^t, \end{aligned} \tag{3.2.8}$$

*where*

$$\begin{aligned} a_0(\boldsymbol{\sigma}^t, \boldsymbol{\tau}) &= \int_P (\frac{1+\nu}{E} \sigma^t_{\alpha\beta} - \frac{\nu}{E} \sigma^t_{\nu\nu} \delta_{\alpha\beta}) \tau_{\alpha\beta}, \\ a_2(\boldsymbol{\sigma}^t, \boldsymbol{\tau}) &= \int_P \left[ \frac{(1+\nu)}{E} \sigma^t_{\alpha 3} \tau_{\alpha 3} - \frac{\nu}{E} (\sigma^t_{33} \tau_{\mu\mu} + \tau_{33} \sigma^t_{\mu\mu}) \right], \\ a_4(\boldsymbol{\sigma}^t, \boldsymbol{\tau}) &= \int_P \frac{1}{E} \sigma^t_{33} \tau_{33}, \\ b(\boldsymbol{\tau}, \vec{v}) &= -\int_P \tau_{ij} \frac{\partial v_j}{\partial x_i}. \end{aligned} \tag{3.2.9}$$

*Proof.* By (1.12),

$$A\boldsymbol{\sigma} = \frac{1+\nu}{E} \boldsymbol{\sigma} - \frac{\nu}{E} \operatorname{tr} \boldsymbol{\sigma} \boldsymbol{\delta}.$$

Let $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ scale to $\boldsymbol{\sigma}^t$ and $\boldsymbol{\tau}^t$ according to (3.2.2). Then

$$\begin{aligned} \frac{1}{t} \int_{P^t} A\boldsymbol{\sigma} : \boldsymbol{\tau} = \int_P \Bigg\{ & \left[ \frac{1+\nu}{E} \sigma^t_{\alpha\beta} - \frac{\nu}{E} \left( \sigma^t_{\nu\nu} + t^2 \sigma^t_{33} \right) \right] \delta_{\alpha\beta} \tau^t_{\alpha\beta} \\ & + t^2 \frac{1+\nu}{E} \sigma^t_{\alpha 3} \tau^t_{\alpha 3} + t^2 \left[ t^2 \frac{1+\nu}{E} \sigma^t_{33} - \frac{\nu}{E} \left( \sigma^t_{\mu\mu} + t^2 \sigma^t_{33} \right) \right] \tau^t_{33} \Bigg\}. \end{aligned}$$

Collecting the like power terms of $t$, we obtain

$$\frac{1}{t} \int_{P^t} A\boldsymbol{\sigma} : \boldsymbol{\tau} = a_0(\boldsymbol{\sigma}^t, \boldsymbol{\tau}^t) + t^2 a_2(\boldsymbol{\sigma}^t, \boldsymbol{\tau}^t) + t^4 a_4(\boldsymbol{\sigma}^t, \boldsymbol{\tau}^t),$$

where $a_0$, $a_2$ and $a_4$ are given by (3.2.9). Let $\boldsymbol{\sigma}$ and $\vec{v}$ scale to $\boldsymbol{\sigma}^t$ and $\vec{v^t}$ according to (3.2.1), we obtain

$$\frac{1}{t} \int_{P^t} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\vec{v}) = \int_P \boldsymbol{\sigma}^t : \boldsymbol{\varepsilon}(\vec{v^t}).$$

Finally, we have

$$\frac{1}{t}\int_{\Gamma^t} f(\vec{v}\cdot\vec{s}) = \int_{\Gamma} f^t(\vec{v^t}\cdot\vec{s}).$$

Using $\boldsymbol{\tau}$ and $\vec{v}$ instead of $\boldsymbol{\tau}^t$ and $\vec{v^t}$ for simplicity. The lemma then follows from (3.1.3). $\square$

## Sec 3.3. An Auxiliary Problem

Our goal now is to obtain asymptotic estimates for the scaled corrector functions $\boldsymbol{\sigma}^t$ and $\vec{u}^t$. Because this is too difficult to do directly, in this section we shall introduce a simplified auxiliary problem for which, as we shall see later, the asymptotic behavior is the same to the lowest order.

In order to define the auxiliary problem, we need to introduce boundary-fitted coordinates in a neighborhood of the lateral boundary $\Gamma$. Let $L$ be a positive number less than half the smallest radius of curvature of $\partial\Omega$, and let $Q$ be the subset of $P$ consisting of points within distance $L$ of $\Gamma$. If $\vec{z}(\theta)$ is a parameterization of the curve $\partial\Omega\times\{0\}$ by arclength, then the mapping

$$(\xi,\theta,x_3)\mapsto \vec{z}(\theta)-\xi\vec{n}+x_3\vec{e}_3$$

defines a diffeomorphism of $\hat{Q}:=(0,L)\times\mathbb{R}/S\times(-1/2,1/2)$ onto $Q$. Here $S$ is the arclength of $\partial\Omega$ and $\mathbb{R}/S$ denotes the real numbers modulo $S$.

The boundary of $Q$ consists of the top and bottom surfaces $\Psi^{\pm}:=Q\cap\Omega_{\pm}$, the outer lateral boundary $\Gamma$, and the inner lateral boundary $\Gamma_L$. These have simple expressions in boundary-fitted coordinates:

$$\hat{\Psi}^{\pm}=(0,L)\times\mathbb{R}/S\times\{\pm1/2\},$$

$$\hat{\Gamma}=\{0\}\times\mathbb{R}/S\times(-1/2,1/2),\quad \hat{\Gamma}_L=\{L\}\times\mathbb{R}/S\times(-1/2,1/2).$$

If $\vec{x}_3 \in Q$, we denote by $\hat{\vec{x}}$ the corresponding point in $\hat{Q}$. If $f$ is a function on $Q$ we define the associated function on $\hat{Q}$ by

$$\hat{f}(\hat{\vec{x}}) = f(\vec{x}).$$

The vector fields $\vec{n}$ and $\vec{s}$ can be extended from $\Gamma$ to all of $Q$ by assigning to each point of $Q$ the value of these vector fields at the unique point of $\Gamma$ nearest the given point. In terms of boundary-fitted coordinates, $\vec{n}$ and $\vec{s}$ are extended from the surface $\xi = 0$ by taking them to be independent of $\xi$. It is easy to check that

$$\frac{\partial\theta}{\partial x_\alpha} = \frac{s_\alpha}{1 - \xi/R(\theta)}, \quad \frac{\partial\xi}{\partial x_\alpha} = -n_\alpha \quad \alpha = 1, 2,$$

and

$$\widehat{\frac{\partial f}{\partial x_\alpha}} = \frac{\partial \hat{f}}{\partial\theta}\frac{\partial\theta}{\partial x_\alpha} + \frac{\partial \hat{f}}{\partial\xi}\frac{\partial\xi}{\partial x_\alpha}, \quad \alpha = 1, 2, \quad \widehat{\frac{\partial f}{\partial x_3}} = \frac{\partial \hat{f}}{\partial x_3}.$$

Change of variable in integration gives

$$\int_Q f = \int_{\hat{Q}} \hat{f} J = \int_{-1/2}^{1/2} \int_0^S \int_0^L \hat{f}\hat{J}\, d\xi d\theta dx_3. \tag{3.3.1}$$

where $\hat{J} = 1 - \xi/R(\theta)$ is the Jacobi determinant. In the sequel we shall usually omit the circumflex from the notation and rely on the context to distinguish between the functions $f$ and $\hat{f}$.

From the boundary-fitted coordinate system we have an orthogonal frame $(-\vec{n}, \vec{s}, \vec{x}_3)$ defined at each point of $Q$. We shall use this frame to express vector and tensor fields. Thus if $\vec{v}$ is a vector field defined on $Q$, we can write

$$\vec{v} = -v_n\vec{n} + v_s\vec{s} + v_3\vec{e}_3$$

where

$$v_n = -\vec{v} \cdot \vec{n}, \quad v_s = \vec{v} \cdot \vec{s}, \quad v_3 = \vec{v} \cdot \vec{e}_3. \tag{3.3.2}$$

Similarly, if $\boldsymbol{\tau}$ is a symmetric tensor field on $Q$, then

$$\boldsymbol{\tau} = \tau_{nn}\vec{n}\vec{n}^T + \tau_{ns}(\vec{s}\vec{n}^T + \vec{n}\vec{s}^T) + \tau_{ss}\vec{s}\vec{s}^T + \tau_{n3}(\vec{n}\vec{e}_3^T + \vec{e}_3\vec{n}^T) + \tau_{s3}(\vec{s}\vec{e}_3^T + \vec{e}_3\vec{s}^T) + \tau_{33}\vec{e}_3\vec{e}_3^T \tag{3.3.3}$$

where

$$\tau_{nn} = \vec{n}^T\boldsymbol{\tau}\vec{n}, \quad \tau_{ns} = \vec{s}^T\boldsymbol{\tau}\vec{n}, \quad \tau_{ss} = \vec{s}^T\boldsymbol{\tau}\vec{s},$$
$$\tau_{n3} = \vec{n}^T\boldsymbol{\tau}\vec{e}_3, \quad \tau_{s3} = \vec{s}^T\boldsymbol{\tau}\vec{e}_3, \quad \tau_{33} = \vec{e}_3^T\boldsymbol{\tau}\vec{e}_3.$$

Next we restate the differential equations (3.2.5), (3.2.7) determining the scaled boundary corrector in terms of boundary-fitted coordinates. Using the fact that

$$\frac{\partial\vec{n}}{\partial\xi} = \frac{\partial\vec{s}}{\partial\xi} = 0, \quad \frac{\partial\vec{n}}{\partial\theta} = \frac{\vec{s}}{R}, \quad \frac{\partial\vec{s}}{\partial\theta} = -\frac{\vec{n}}{R},$$

we get

$$\frac{\partial\sigma^t_{nn}}{\partial\xi} + \frac{1}{1-\xi/R}\frac{\partial\sigma^t_{ns}}{\partial\theta} + \frac{\partial\sigma^t_{n3}}{\partial x_3} = 0,$$
$$\frac{\partial\sigma^t_{ns}}{\partial\xi} + \frac{1}{1-\xi/R}\frac{\partial\sigma^t_{ss}}{\partial\theta} + \frac{\partial\sigma^t_{s3}}{\partial x_3} = 0,$$
$$\frac{\partial\sigma^t_{n3}}{\partial\xi} + \frac{1}{1-\xi/R}\frac{\partial\sigma^t_{s3}}{\partial\theta} + \frac{\partial\sigma^t_{33}}{\partial x_3} = 0,$$

and

$$\sigma^t_{nn} = \frac{E}{2(1+\nu)}\left\{2\frac{\partial u^t_n}{\partial\xi} + \frac{2\nu}{1-2\nu}\left[\frac{\partial u^t_n}{\partial\xi} + \frac{1}{t^2}\frac{\partial u^t_3}{\partial x_3} + \frac{1}{1-\xi/R}\left(\frac{\partial u^t_s}{\partial\theta} - \frac{1}{R}u^t_n\right)\right]\right\},$$
$$\sigma^t_{ns} = \frac{E}{2(1+\nu)}\left[\frac{\partial u^t_s}{\partial\xi} + \frac{1}{1-\xi/R(\theta)}\left(\frac{\partial u^t_n}{\partial\theta} + \frac{1}{R}u^t_s\right)\right],$$
$$\sigma^t_{ss} = \frac{E}{(1+\nu)(1-2\nu)}\left[\frac{\nu}{t^2}\left(\frac{\partial u^t_3}{\partial x_3} + t^2\frac{\partial u^t_n}{\partial\xi}\right) + \frac{1-\nu}{1-\xi/R}\left(\frac{\partial u^t_s}{\partial\theta} - \frac{1}{R}u^t_n\right)\right],$$
$$\sigma^t_{n3} = \frac{E}{2(1+\nu)t^2}\left(\frac{\partial u^t_3}{\partial\xi} + \frac{\partial u^t_n}{\partial x_3}\right),$$
$$\sigma^t_{s3} = \frac{E}{2(1+\nu)t^2}\left(\frac{\partial u^t_s}{\partial x_3} + \frac{1}{1-\xi/R}\frac{\partial u^t_3}{\partial\theta}\right),$$
$$\sigma^t_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)t^4}\left\{\frac{\partial u^t_3}{\partial x_3} + \frac{t^2\nu}{1-\nu}\left[\frac{\partial u^t_n}{\partial\xi} + \frac{1}{1-\xi/R}\left(\frac{\partial u^t_s}{\partial\theta} - \frac{1}{R}u^t_n\right)\right]\right\}.$$

To define the auxiliary problem, we alter these differential equations, by suppressing terms that arise from differentiation with respect to $\theta$. Thus we shall define a tensor field $\boldsymbol{\rho}$ and a vector field $\vec{y}$ on $Q$ satisfying

$$\begin{aligned}
\rho_{nn} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)t^2}\left(\frac{\nu}{1-\nu}\frac{\partial y_3}{\partial x_3} + t^2\frac{\partial y_n}{\partial \xi}\right),\\
\rho_{ns} &= \frac{E}{2(1+\nu)}\frac{\partial y_s}{\partial \xi},\\
\rho_{ss} &= \frac{E\nu}{(1+\nu)(1-2\nu)t^2}\left(\frac{\partial y_3}{\partial x_3} + t^2\frac{\partial y_n}{\partial \xi}\right),\\
\rho_{n3} &= \frac{E}{2(1+\nu)t^2}\left(\frac{\partial y_3}{\partial \xi} + \frac{\partial y_n}{\partial x_3}\right),\\
\rho_{s3} &= \frac{E}{2(1+\nu)t^2}\frac{\partial y_s}{\partial x_3},\\
\rho_{33} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)t^4}\left(\frac{\partial y_3}{\partial x_3} + \frac{t^2\nu}{1-\nu}\frac{\partial y_n}{\partial \xi}\right),
\end{aligned} \tag{3.3.4}$$

and

$$\begin{aligned}
\frac{\partial \rho_{nn}}{\partial \xi} + \frac{\partial \rho_{n3}}{\partial x_3} &= 0,\\
\frac{\partial \rho_{ns}}{\partial \xi} + \frac{\partial \rho_{s3}}{\partial x_3} &= 0,\\
\frac{\partial \rho_{n3}}{\partial \xi} + \frac{\partial \rho_{33}}{\partial x_3} &= 0.
\end{aligned} \tag{3.3.5}$$

The boundary conditions on $\boldsymbol{\rho}$ and $\vec{y}$ are

$$\begin{aligned}
&\rho_{n3} = 0, \quad \rho_{s3} = 0, \quad \rho_{33} = 0 && && \text{on } \Psi^+ \cup \Psi,\\
&\rho_{nn} = 0, \quad \rho_{sn} = -f, \quad \mu = h_3 && \text{on } \Gamma_0 && \text{soft case},\\
&\rho_{nn} = 0, \quad y_s = h_s, \quad y_3 = h_3 && \text{on } \Gamma_0 && \text{hard case},\\
&y_n = 0, \quad \rho_{ns} = 0, \quad y_3 \text{ is constant} && && \text{on } \Gamma_L,\\
&\int_{\Gamma_L} \rho_{n3} = 0.
\end{aligned} \tag{3.3.6}$$

We will discuss the existence and uniqueness of the problem in the next section.

The solution $(\boldsymbol{\rho}, \vec{y})$ satisfies a weak formulation which we will use in Section 3.8 in the error analysis. For this purpose we introduces the following spaces.

$$\boldsymbol{\Sigma}_Q = \left\{ \boldsymbol{\tau} \mid \tau_{ij} \in L^2(Q), \tau_{ij} = \tau_{ji} \right\}, \tag{3.3.7}$$

$$\boldsymbol{V}_Q = \left\{ \vec{v} \mid v_i \in H^1(Q),\ \vec{v} \cdot \vec{e}_3 = 0 \text{ on } \Gamma_0,\ \vec{v} \cdot \vec{n} = 0,\ \vec{v} \cdot \vec{e}_3 = \text{constant on } \Gamma_L \right\} \quad \text{soft case},$$

$$\boldsymbol{V}_Q = \left\{ \vec{v} \mid v_i \in H^1(Q),\ \vec{v} \cdot \vec{e}_3 = \vec{v} \cdot \vec{s} = 0 \text{ on } \Gamma_0,\ \vec{v} \cdot \vec{n} = 0,\ \vec{v} \cdot \vec{e}_3 = \text{constant on } \Gamma_L \right\} \quad \text{hard case}.$$

**Lemma 3.3.1.** *Let $(\boldsymbol{\rho}, \vec{y})$ be a solution to (3.3.4)–(3.3.6). Then $(\boldsymbol{\rho}, \vec{y}) \in \boldsymbol{\Sigma}_\beta \times [H^1(Q)]^3$, $\vec{y}$ satisfies (3.3.6) on $\Gamma_0 \cup \Gamma_L$, and*

$$A_0^Q(\boldsymbol{\rho}, \boldsymbol{\tau}) + t^2 A_2^Q(\boldsymbol{\rho}, \boldsymbol{\tau}) + t^4 A_4^Q(\boldsymbol{\rho}, \boldsymbol{\tau}) + B^Q(\boldsymbol{\tau}, \vec{y}) = 0 \textit{ for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma}_Q, \tag{3.3.8}$$

$$B^Q(\boldsymbol{\rho}, \vec{v}) = \begin{cases} \int_\Gamma f^t(\vec{v} \cdot \vec{s}) & \textit{soft case} \\ 0 & \textit{hard case} \end{cases} \qquad \textit{for all } \vec{v} \in \boldsymbol{V}_Q, \tag{3.3.9}$$

*where*

$$\begin{aligned}
A_0^Q(\boldsymbol{\rho}, \boldsymbol{\tau}) &= \int_Q \left( \frac{1+\nu}{E} \rho_{\alpha\beta} - \frac{\nu}{E} \rho_{\nu\nu} \delta_{\alpha\beta} \right) \tau_{\alpha\beta} J^{-1}, \\
A_2^Q(\boldsymbol{\rho}, \boldsymbol{\tau}) &= \int_Q \left[ \frac{(1+\nu)}{E} \rho_{\alpha 3} \tau_{\alpha 3} - \frac{\nu}{E} (\rho_{33} \tau_{\mu\mu} + \tau_{33} \rho_{\mu\mu}) \right] J^{-1}, \\
A_4^Q(\boldsymbol{\rho}, \boldsymbol{\tau}) &= \int_Q \frac{1}{E} \rho_{33} \tau_{33} J^{-1}, \\
B^Q(\boldsymbol{\tau}, \vec{v}) &= \int_Q \left( \tau_{nn} \frac{\partial v_n}{\partial \xi} + \tau_{ns} \frac{\partial v_s}{\partial \xi} + \tau_{n3} \frac{\partial v_3}{\partial \xi} + \tau_{n3} \frac{\partial v_n}{\partial x_3} \right. \\
&\qquad \left. + \tau_{s3} \frac{\partial v_s}{\partial x_3} + \tau_{33} \frac{\partial v_3}{\partial x_3} \right) J^{-1}.
\end{aligned} \tag{3.3.10}$$

*Proof.* Multiply the three equations in (3.3.5) by $v_n$, $v_s$, and $v_3$, respectively, and integrate over $Q$. Integrating by parts and using the boundary conditions in (3.3.6) gives (3.3.9).

Associate to a tensor $\boldsymbol{\mu}$ the scaled version

$$\tilde{\mu}_{nn} = \mu_{nn}, \quad \tilde{\mu}_{ns} = \mu_{ns}, \quad \tilde{\mu}_{ss} = \mu_{ss},$$
$$\tilde{\mu}_{n3} = t\mu_{n3}, \quad \tilde{\mu}_{s3} = t\mu_{s3}, \quad \tilde{\mu}_{33} = t^2\mu_{33}.$$

Further, define a tensor $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\vec{y})$ by

$$\gamma_{nn} = \frac{\partial y_n}{\partial \xi}, \qquad \gamma_{ns} = \frac{1}{2}\frac{\partial y_s}{\partial \xi}, \quad \gamma_{n3} = \frac{1}{2t}\left(\frac{\partial y_3}{\partial \xi} + \frac{\partial y_n}{\partial x_3}\right),$$
$$\gamma_{s3} = \frac{1}{2t}\frac{\partial y_s}{\partial x_3}, \quad \gamma_{33} = \frac{1}{t^2}\frac{\partial y_3}{\partial x_3}, \quad \gamma_{ss} = 0.$$

Then

$$A_0^Q(\boldsymbol{\rho}, \boldsymbol{\tau}) + t^2 A_2^Q(\boldsymbol{\rho}, \boldsymbol{\tau}) + t^4 A_4^Q(\boldsymbol{\rho}, \boldsymbol{\tau}) = \int_Q A\tilde{\boldsymbol{\rho}} : \tilde{\boldsymbol{\tau}} J^{-1},$$

and

$$B^Q(\boldsymbol{\tau}, \vec{y}) = \int_Q \tilde{\boldsymbol{\tau}} : \boldsymbol{\gamma}(\vec{y}) J^{-1}.$$

(The fourth order tensor $A$ is given in (1.12)). Therefore the equation (3.3.6) holds if and only if $A\tilde{\boldsymbol{\rho}} = \boldsymbol{\gamma}(\vec{y})$. This last equation follows directly from the definition of these tensors. $\square$

Note that $A_i^Q$ differs from $a_i$ in two ways: restriction of the domain to $Q$ and multiplication of $J^{-1}$, If we let $\chi_Q$ be the characteristic function of $Q$, then

$$A_i^Q(J\boldsymbol{\rho}, \boldsymbol{\tau}) = a_i(\chi_Q \boldsymbol{\rho}, \boldsymbol{\tau}), \qquad i = 0, 2, 4. \tag{3.3.11}$$

Note also that $B^Q(\boldsymbol{\tau}, \vec{v})$ differs from $b(\boldsymbol{\tau}, \vec{v})$ in three ways: restriction of the domain to $Q$, suppression of terms involving tangential derivatives, and multiplication of $J^{-1}$. In fact,

$$b(\chi_Q \boldsymbol{\tau}, \vec{v}) = \int_Q \left\{ \tau_{nn}\frac{\partial v_n}{\partial \xi} + \tau_{ns}\left[\frac{\partial v_s}{\partial \xi} + \frac{1}{1-\xi/R}\left(\frac{\partial v_n}{\partial \theta} + \frac{1}{R}v_s\right)\right] + \tau_{n3}\frac{\partial v_3}{\partial \xi} \right.$$
$$\left. + \tau_{n3}\frac{\partial v_n}{\partial x_3} + \tau_{s3}\left(\frac{\partial v_s}{\partial x_3} + \frac{1}{1-\xi/R}\frac{\partial v_3}{\partial \theta}\right) + \tau_{ss}\left(\frac{\partial v_s}{\partial \theta} - \frac{1}{R}v_n\right) + \tau_{33}\frac{\partial v_3}{\partial x_3} \right\}$$

Thus, let

$$\boldsymbol{\beta} = \frac{1}{1-\xi/R}\left[\frac{1}{2}\left(\frac{\partial v_n}{\partial\theta}+\frac{1}{R}v_s\right)(\vec{n}\vec{s}^T+\vec{s}\vec{n}^T)+\left(\frac{\partial v_s}{\partial\theta}-\frac{1}{R}v_n\right)\vec{s}\vec{s}^T\right.$$
$$\left.+\frac{1}{2}\frac{\partial y_3}{\partial\theta}(\vec{e}_3\vec{s}^T+\vec{s}\vec{e}_3^T)\right],$$

then it follows that

$$B^\beta(J\boldsymbol{\tau},\vec{v}) = b(\chi_Q\boldsymbol{\tau},\vec{v}) - \int_Q \boldsymbol{\tau}:\boldsymbol{\beta}. \tag{3.3.12}$$

Substitute (3.3.4) into (3.3.5), (3.3.6), we obtain explicitly the following differential equations and boundary conditions in terms of only the displacement variables:

$$\begin{aligned}
&2t^2(1-\nu)\frac{\partial^2 y_n}{\partial\xi^2}+\frac{\partial^2 y_3}{\partial x_3\partial\xi}+(1-2\nu)\frac{\partial^2 y_n}{\partial x_3^2}=0 && \text{in } Q,\\
&2(1-\nu)\frac{\partial^2 y_3}{\partial x_3^2}+t^2\frac{\partial^2 y_n}{\partial x_3\partial\xi}+t^2(1-2\nu)\frac{\partial^2 y_3}{\partial\xi^2}=0 && \text{in } Q,\\
&t^2\frac{\partial^2 y_s}{\partial\xi^2}+\frac{\partial^2 y_s}{\partial x_3^2}=0 && \text{in } Q.
\end{aligned} \tag{3.3.13}$$

with the following boundary condition on top and bottom surfaces:

$$\begin{aligned}
&\frac{\partial y_3}{\partial\xi}+\frac{\partial y_n}{\partial x_3}=0 && \text{on } \Psi^+\cup\Psi^-,\\
&\frac{\nu t^2}{1-\nu}\frac{\partial y_n}{\partial\xi}+\frac{\partial y_3}{\partial x_3}=0 && \text{on } \Psi^+\cup\Psi^-,\\
&\frac{\partial y_s}{\partial x_3}=0 && \text{on } \Psi^+\cup\Psi^-.
\end{aligned}$$

On the lateral boundary $\Gamma_0$, the boundary conditions are as follows:

$$\begin{aligned}
&t^2\frac{\partial y_n}{\partial\xi}+\frac{\nu}{1-\nu}\frac{\partial y_3}{\partial x_3}=0, \quad y_3=h_3, \quad \frac{\partial y_s}{\partial\xi}=-\frac{2(1+\nu)}{E}f; && \text{soft case}\\
&t^2\frac{\partial y_n}{\partial\xi}+\frac{\nu}{1-\nu}\frac{\partial y_3}{\partial x_3}=0, \quad y_3=h_3, \quad y_s=h_s; && \text{hard case}
\end{aligned}$$

On the lateral boundary $\Gamma_L$, the boundary conditions are as follows:

$$\begin{aligned}
&y_n=0, \quad \frac{\partial y_s}{\partial\xi}=0, \quad y_3 \text{ is constant} \qquad \text{on } \Gamma_L,\\
&\int_{\Gamma_L}\left(\frac{\partial y_n}{\partial x_3}+\frac{\partial y_3}{\partial\xi}\right)=0.
\end{aligned} \tag{3.3.14}$$

## Sec 3.4. Two Separate Problems

The problem in (3.3.13)–(3.3.14) has derivatives with respect to only the variables $\xi$ and $x_3$. This indicates that the variable $\theta$ can be treated as a parameter. Thus we discuss the problem in the two-dimensional domain $\omega_\theta = (0, L)\times(-1/2, 1/2)\times\{\theta\}$ for every $\theta$. For simplicity we write $\omega$ instead of $\omega_\theta$. In the following discussion, although bounds derived for the unknown variables on the domain $\omega$ seem to depend on $\theta$, such dependence is continuous. As the domain of $\theta$ is a compact set, all such bounds are actually independent of $\theta$. From this this section on, we denote $y_n$ by $\psi$, $y_s$ by $\phi$, and $y_3$ by $\mu$.

We denote by $\gamma^\pm$ the horizontal segment $(0, L) \times \{\pm 1/2\}$, and by $\gamma_\alpha$ the vertical segment $\{\alpha\} \times (-1/2, 1/2)$ for $0 \leq \alpha \leq L$.

It is easy to see that the equations in (3.3.13)–(3.3.14) decouple into two problems.

The first problem relates to a Laplace equation on $\omega$, and determines $\phi$:

$$\begin{aligned}
&t^2 \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial x_3^2} = 0 && \text{in } \omega, \\
&\frac{\partial \phi}{\partial x_3} = 0 && \text{on } \gamma^+ \cup \gamma^-, \\
&\frac{\partial \phi}{\partial \xi} = 0 && \text{on } \gamma_L, \\
&\frac{\partial \phi}{\partial \xi} = -\frac{2(1+\nu)}{E} f && \text{on } \gamma_0; \quad \text{soft case}, \\
&\phi = h_s && \text{on } \gamma_0; \quad \text{hard case}.
\end{aligned} \tag{3.4.1}$$

In the soft case, we add the following condition to guarantee that the solution is unique:

$$\int_\omega \phi = 0. \tag{3.4.2}$$

For soft simply supported plate, the compatibility condition is $\int_{\gamma_0} f = 0$.

The second problem relates to a plane elasticity system, and determines $\psi$ and $\mu$:

$$\begin{aligned}
&2t^2(1-\nu)\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\mu}{\partial x_3\partial\xi}+(1-2\nu)\frac{\partial^2\psi}{\partial x_3^2}=0 && \text{in } \omega,\\
&2(1-\nu)\frac{\partial^2\mu}{\partial x_3^2}+t^2\frac{\partial^2\psi}{\partial x_3\partial\xi}+t^2(1-2\nu)\frac{\partial^2\mu}{\partial\xi^2}=0 && \text{in } \omega,\\
&\frac{\partial\mu}{\partial\xi}+\frac{\partial\psi}{\partial x_3}=0, \quad \frac{\nu t^2}{1-\nu}\frac{\partial\psi}{\partial\xi}+\frac{\partial\mu}{\partial x_3}=0 && \text{on } \gamma^+\cup\gamma^-,\\
&t^2\frac{\partial\psi}{\partial\xi}+\frac{\nu}{1-\nu}\frac{\partial\mu}{\partial x_3}=0, \quad \mu=h_3 && \text{on } \gamma_0,\\
&\psi=0, \quad \mu \text{ is constant} && \text{on } \gamma_L,\\
&\int_{\gamma_L}\left(\frac{\partial\mu}{\partial\xi}+\frac{\partial\psi}{\partial x_3}\right)=0.
\end{aligned} \tag{3.4.3}$$

It is easy to see that either (3.4.1) or (3.4.3) has a unique solution.

We need to find the orders of $\phi$, $\psi$, $\mu$ and components of $\boldsymbol{\rho}$ in several norms. The following space is equipped with a norm that treats partial derivatives along different directions separately:

$$H^{s,0}(\omega)=\left\{ v:\omega\mapsto\mathbb{R} \mid v(\cdot,y)\in H^s(0,L),\ \int_{-1/2}^{1/2}\|v(x,\cdot)\|^2_{H^s(0,L)}\,dy<\infty \right\}.$$

## Sec 3.5. Saint Venant's Principle

In this section we discuss Saint Venant's principle for Laplace's equation and the elasticity equations in a two-dimensional strip. Saint Venant's principle describes that under certain conditions, the effect of nonzero boundary data decay quickly away from the boundary. [22], [12], [16], [15], [14].

We also derive bounds for some energy norms. Results obtained in this section will be used in later estimation. The notation is independent of the previous sections.

For any numbers $r_1 < r_2$ let $\omega_{r_1,r_2}$ denote the rectangle $(r_1, r_2) \times (-1/2, 1/2)$. For $r > 0$ we use the simpler notation $\omega_r$ in place of $\omega_{0,r}$. Also we denote by $\gamma_r$ the vertical segment $\{r\} \times (-1/2, 1/2)$, and by $\gamma_r^\pm$ the horizontal segments $(0, r) \times \{\pm 1/2\}$.

**Exponential decay for Laplace's equation in a strip.** In this subsection we consider Laplace's equation

$$\Delta u = 0 \qquad \text{in } \omega_r, \tag{3.5.1}$$

subject to the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \gamma_r^+ \cup \gamma_r^- \cup \gamma_r \tag{3.5.2}$$

on three sides of the strip. On the fourth side we impose either the Dirichlet condition (the hard case)

$$u = h \qquad \text{on } \gamma_0 \tag{3.5.3}$$

or the Neumann condition (the soft case) (the soft case)

$$\frac{\partial u}{\partial n} = g \qquad \text{on } \gamma_0, \tag{3.5.4}$$

where $g$ is a given function on $\gamma_0$ of mean value zero:

$$\int_{\gamma_0} g = 0. \tag{3.5.5}$$

In the case of the Neumann condition, we also normalize the solution by imposing the condition

$$\int_{\omega_r} u = 0. \tag{3.5.6}$$

Given $g$ satisfying (3.5.5), there is clearly a unique solution $u$ to (3.5.1), (3.5.2), and either (3.5.3) or (3.5.4) and (3.5.6). Note that, in view of (3.5.5), equation (3.5.6) holds in both cases. The main result of this subsection is the following exponential decay result.

**Theorem 3.5.1.** *Let $r > 0$ and let $u : \omega_r \to \mathbb{R}$ be the unique solution to one of the boundary value problems described above. Then*

$$\int_{\omega_{r_2,r}} |\underset{\sim}{\operatorname{grad}}\, u|^2 \leq e^{-\lambda(r_2 - r_1)} \int_{\omega_{r_1,r}} |\underset{\sim}{\operatorname{grad}}\, u|^2 \qquad \text{for all } 0 \leq r_1 \leq r_2 \leq r.$$

*Here $\lambda$ is the constant $8\pi^2/(4\pi^2 + 1)$.*

Before turning to the proof, we establish some simple lemmas. The following standard result may be derived from a Fourier series expansion.

**Lemma 3.5.2.** *Let $g \in H^1(I)$ where $I$ is an interval of length 1 and suppose that $\int_I g = 0$. Then*

$$\int_I |g|^2 \leq \frac{1}{4\pi^2} \int_I |g'|^2$$

*(where the prime denotes differentiation along $I$).*

Next we show that the solution $u$ has mean value zero on every vertical segment.

**Lemma 3.5.3.** *Let $r$ and $u$ be as in Theorem 3.5.1. Then*

$$\int_{\gamma_s} u = 0$$

*for all $0 \leq s \leq r$.*

*Proof.* From (3.5.1), Green's theorem, and (3.5.2) we have

$$0 = \int_{\omega_{s,r}} \Delta u = \int_{\partial \omega_{s,r}} \frac{\partial u}{\partial n} = \int_{\gamma_s} \frac{\partial u}{\partial n}.$$

Define $v(x, y) = x$. Again applying (3.5.1), Green's theorem, and (3.5.2), we get

$$0 = \int_{\omega_s} (v\Delta u - u\Delta v) = \int_{\partial \omega_s} \left( v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) = -\int_{\partial \omega_s} u\frac{\partial v}{\partial n} = \int_{\gamma_0} u - \int_{\gamma_s} u.$$

The lemma follows from this equality and (3.5.6). $\square$

We now turn to the proof of Theorem 3.5.1.

*Proof.* For $0 < s < r$, let $E(s) = \int_{\omega_{s,r}} |\underset{\sim}{\operatorname{grad}}\, u|^2$, so $E : (0, r) \to \mathbb{R}$ is positive and decreasing.

From (3.5.1), Green's theorem, and (3.5.2) we have

$$E(s) = \int_{\partial\omega_{s,r}} u \frac{\partial u}{\partial n} = \int_{\gamma_s} u \frac{\partial u}{\partial n} \le \frac{1}{2} \int_{\gamma_s} |u|^2 + \frac{1}{2} \int_{\gamma_s} \left| \frac{\partial u}{\partial n} \right|^2,$$

where we have used the arithmetic-geometric mean inequality in the last step. In view of Lemma 3.5.3 we may apply Lemma 3.5.2 to get

$$E(s) \le \left( \frac{1}{2} + \frac{1}{8\pi^2} \right) \int_{\gamma_s} |\underset{\sim}{\operatorname{grad}}\, u|^2,$$

or

$$E(s) \le -\frac{1}{\lambda} E'(s).$$

Rewriting this as $E'(s)/E(s) \le -\lambda$, integrating from $r_1$ to $r_2$, and exponentiating then gives

$$E(r_2) \le e^{-\lambda(r_2 - r_1)} E(r_1)$$

as desired. □

Note that from Lemmas 2 and 3 we easily obtain the estimate

$$\|u\|^2_{L^2(\omega_{s,r})} \le \frac{1}{4\pi^2} \|\underset{\sim}{\operatorname{grad}}\, u\|^2_{L^2(\omega_{s,r})}.$$

Therefore we obtain exponential decay of the $H^1$ norm as a corollary of Theorem 3.5.1.

**Theorem 3.5.4.** *Under the conditions of Theorem 3.5.1*

$$\|u\|^2_{H^1(\omega_{r_2,r})} \le \left( 1 + \frac{1}{4\pi^2} \right) e^{-\lambda(r_2 - r_1)} \int_{\omega_{r_1,r}} |\underset{\sim}{\operatorname{grad}}\, u|^2 \qquad \textit{for all } 0 \le r_1 \le r_2 \le r.$$

**Exponential decay for two-dimensional elasticity equation in a strip.** Let $A$ be a symmetric positive definite tensor on symmetric $2 \times 2$ matrices. We consider the following two-dimensional elasticity equations in this subsection:

$$\underset{\approx}{\sigma} = A\underset{\approx}{\varepsilon}(\underset{\sim}{u}) \qquad \text{in } \omega_r, \tag{3.5.7}$$

$$\underset{\sim}{\text{div}}\underset{\approx}{\sigma} = \underset{\sim}{0} \qquad \text{in } \omega_r, \tag{3.5.8}$$

subject to the homogeneous Neumann boundary condition

$$\underset{\approx}{\sigma}\underset{\sim}{n} = \underset{\sim}{0} \qquad \text{on } \gamma_r^+ \cup \gamma_r^-, \tag{3.5.9}$$

on the top and bottom sides of the strip. On one side we impose

$$\underset{\sim}{n}^T \underset{\approx}{\sigma}\underset{\sim}{n} = 0, \quad \underset{\sim}{u} \cdot \underset{\sim}{s} = h \qquad \text{on } \gamma_0. \tag{3.5.10}$$

On the other side we impose

$$\begin{aligned} &\underset{\sim}{u} \cdot \underset{\sim}{n} = 0, \quad \underset{\sim}{u} \cdot \underset{\sim}{s} \text{ is constant} \qquad \text{on } \gamma_r, \\ &\int_{\gamma_r} \underset{\sim}{s}^t \underset{\approx}{\sigma}\underset{\sim}{n} = 0. \end{aligned} \tag{3.5.11}$$

Define the space for displacement

$$\underset{\sim}{V}_h = \left\{ \underset{\sim}{v} \in H^1(\Omega_r) | \underset{\sim}{v} \cdot \underset{\sim}{s} = h \text{ on } \gamma_0, \underset{\sim}{v} \cdot \underset{\sim}{n} = 0, \underset{\sim}{v} \cdot \underset{\sim}{s} \text{ is constant on } \gamma_r \right\}.$$

The weak formulation of (3.5.7)–(3.5.11) is the following:

Find $\underset{\sim}{u} \in \underset{\sim}{V}_h$ such that

$$\int_{\Omega_r} [A\underset{\approx}{\varepsilon}(\underset{\sim}{u})] : \underset{\approx}{\varepsilon}(\underset{\sim}{v}) = 0 \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{V}_0. \tag{3.5.12}$$

Note that $\left\{ \int_{\Omega_r} [A\underset{\approx}{\varepsilon}(\underset{\sim}{u})] : \underset{\approx}{\varepsilon}(\underset{\sim}{u}) \right\}^{1/2}$ is a norm on $\underset{\sim}{V}_0$, so the equation (3.5.12) has a unique solution. The main result of this subsection is the following exponential decay result.

**Theorem 3.5.5.** *Let $r > 0$ and let $\underset{\sim}{u} \in \underset{\sim}{V}_h$ be the unique solution to* (3.5.12). *Then there exist positive constants $\lambda$ and $C$ independent of $r$ such that*

$$\int_{\omega_{r_2,r}} |\underset{\approx}{\sigma}|^2 \leq Ce^{-\lambda(r_2-r_1)} \int_{\omega_{r_1,r}} |\underset{\approx}{\sigma}|^2 \qquad \text{for all } r_1 \geq 0,\ r_1+1 \leq r_2 \leq r,$$

$$\int_{\omega_{r_2,r}} |\underset{\approx}{\varepsilon}(\underset{\sim}{u})|^2 \leq Ce^{-\lambda(r_2-r_1)} \int_{\omega_{r_1,r}} |\underset{\approx}{\varepsilon}(\underset{\sim}{u})|^2 \qquad \text{for all } r_1 \geq 0,\ r_1+1 \leq r_2 \leq r.$$

Before turning to the proof, we establish a simple lemma and corollary.

**Lemma 3.5.6.** *Let $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ be the solution to* (3.5.7)–(3.5.11). *Then*

$$\int_{\gamma_s} \underset{\approx}{\sigma}\underset{\sim}{n} = 0, \quad \int_{\gamma_s} y\underset{\sim}{n}^T \underset{\approx}{\sigma}\underset{\sim}{n} = 0 \qquad \text{for all } 0 \leq s \leq r.$$

*Proof.* First, let $\underset{\sim}{v} = (0,1)$. From Green's theorem, (3.5.8), (3.5.9), and (3.5.11), for any $0 \leq s < r$, we obtain

$$0 = \int_{\omega_{s,r}} \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{v}) = \int_{\partial\omega_{s,r}} (\underset{\approx}{\sigma}\underset{\sim}{n}) \cdot \underset{\sim}{v} - \int_{\omega_{s,r}} (\underset{\sim}{\mathrm{div}}\underset{\approx}{\sigma}) \cdot \underset{\sim}{v} = \int_{\gamma_s} \underset{\sim}{s}^T \underset{\approx}{\sigma}\underset{\sim}{n}. \tag{3.5.13}$$

Next, let $\underset{\sim}{v} = (1,0)$. From Green's theorem, (3.5.8), (3.5.9) and (3.5.10), for any $0 < s \leq r$, we obtain

$$0 = \int_{\omega_s} \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{v}) = \int_{\partial\omega_s} (\underset{\approx}{\sigma}\underset{\sim}{n}) \cdot \underset{\sim}{v} - \int_{\omega_s} (\underset{\sim}{\mathrm{div}}\underset{\approx}{\sigma}) \cdot \underset{\sim}{v} = \int_{\gamma_s} \underset{\sim}{n}^T \underset{\approx}{\sigma}\underset{\sim}{n}.$$

Finally take $\underset{\sim}{v} = (y, -x)$. From Green's theorem, (3.5.8), (3.5.9) and (3.5.10), for any $0 < s \leq r$, we obtain

$$\begin{aligned} 0 &= \int_{\omega_s} \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{v}) = \int_{\partial\omega_s} (\underset{\approx}{\sigma}\underset{\sim}{n}) \cdot \underset{\sim}{v} - \int_{\omega_s} (\underset{\sim}{\mathrm{div}}\underset{\approx}{\sigma}) \cdot \underset{\sim}{v} \\ &= \int_{\gamma_s} y\underset{\sim}{n}^T \underset{\approx}{\sigma}\underset{\sim}{n} - s\int_{\gamma_s} \underset{\sim}{s}^t \underset{\approx}{\sigma}\underset{\sim}{n} = \int_{\gamma_s} y\underset{\sim}{n}^T \underset{\approx}{\sigma}\underset{\sim}{n}. \end{aligned}$$

The last equality follows from (3.5.13). $\square$

**Corollary 3.5.7.** *If* $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ *solves* (3.5.7)-(3.5.11), *then*

$$\int_{\gamma_s} (\underset{\approx}{\sigma}\underset{\sim}{n}) \cdot \underset{\sim}{m} = 0 \qquad \textit{for all } 0 \leq s \leq r, \quad \textit{for all } \underset{\sim}{m} \in \mathcal{N},$$

*where* $\mathcal{N} = \{ a + by, c - bx \mid a, b, c \in \mathbb{R} \}$. □

We now turn to the proof of Theorem 3.5.5.

*Proof.* We set

$$E(s) = \int_{\omega_{s,r}} \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{u}).$$

For each $s \geq 0$, denote by $\mathcal{N}_s$ the space of rigid motions in the domain $\omega_{s,s+1} = (s, s+1) \times (-1/2, 1/2)$. It is well known that any rigid motion takes the form $(a + cy, b - cx)$ for some constants $a$, $b$, $c$, and so $\mathcal{N}_s$ is a space of dimension 3. When $s \leq r - 1$ fixed, let $\underset{\sim}{\bar{u}}$ be such that $\underset{\sim}{u} - \underset{\sim}{\bar{u}} \in \mathcal{N}_s$ and $\underset{\sim}{\bar{u}}$ is orthogonal to $\mathcal{N}_s$ in $L^2(\omega_{s,s+1})$. It follows from Korn's inequality that there is a constant C independent of $r$ and $s$ such that

$$\int_{\omega_{s,s+1}} |\underset{\sim}{\bar{u}}|^2 \leq C \int_{\omega_{s,s+1}} |\underset{\approx}{\varepsilon}(\underset{\sim}{u})|^2.$$

Let

$$Q(s) = \int_s^{s+1} E(p)\, dp \quad \text{for all } \ 0 \leq s \leq r - 1.$$

From Green's theorem, (3.5.8), (3.5.9), (3.5.11) and Corollary 3.5.7, for any $s$, we have

$$\begin{aligned} Q(s) &= \int_s^{s+1} \int_{\omega_{p,r}} \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{u}) = \int_s^{s+1} \int_{\gamma_p \cup \gamma_r} (\underset{\approx}{\sigma}\underset{\sim}{n}) \cdot \underset{\sim}{u} = \int_s^{s+1} \int_{\gamma_p} (\underset{\approx}{\sigma}\underset{\sim}{n}) \cdot \underset{\sim}{\bar{u}} \\ &\leq \frac{1}{2} \int_{\omega_{s,s+1}} \left( |\underset{\approx}{\sigma}|^2 + |\underset{\sim}{\bar{u}}|^2 \right) \leq C \int_{\omega_{s,s+1}} |\underset{\approx}{\sigma}|^2 \leq \frac{1}{\lambda} \int_{\omega_{s,s+1}} \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{u}) \end{aligned}$$

where $\lambda$ is some positive constant. On the other hand,

$$\int_{\omega_{s,s+1}} \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{u}) = E(s) - E(s+1) = -Q'(s).$$

Thus,

$$Q(s) \le -\frac{1}{\lambda} Q'(s).$$

Rewriting this as $Q'(s)/Q(s) \le -\lambda$, integrating from $r_1$ to $r_2$, and exponentiating then gives

$$Q(r_2) \le e^{-\lambda(r_2-r_1)} Q(r_1) \quad \text{for all} \;\; 0 \le r_1 \le r_2 \le r-1.$$

Note that $E(s)$ is a decreasing function, so

$$E(s+1) \le Q(s) \le E(s) \quad \text{for all} \;\; 0 \le s \le r-1.$$

It follows that

$$E(r_2) \le e^{\lambda} e^{-\lambda(r_2-r_1)} E(r_1) \quad \text{for all} \;\; r_1 \ge 0, r_1+1 \le r_2 \le r.$$

The theorem follows easily. □

**Energy norm bounds.** In this subsection, we show that the energy norm of the solution to either the Laplace equation or the elasticity equations can be bounded in terms of the lateral boundary data on $\gamma_0$ uniformly in $r$.

**Theorem 3.5.8.** *Let $u$, $h$, $g$ satisfy* (3.5.1)–(3.5.6). *There exist constants $C_1$, $C_2$ independent of $r \ge 1$ such that*

$$C_1 \|g\|_{H^{-1/2}(\gamma_0)} \le \|u\|_{H^1(\omega_r)} \le C_2 \|g\|_{H^{-1/2}(\gamma_0)} \qquad \textit{soft case},$$

$$C_1 \|h\|_{H^{1/2}(\gamma_0)} \le \|u\|_{H^1(\omega_r)} \le C_2 \|h\|_{H^{1/2}(\gamma_0)} \qquad \textit{hard case}.$$

*Proof.* We first prove the lower bounds. For the hard case, since $u = h$ on $\gamma_0$, by trace theorem, there exists a positive constant $C_1'$ independent of $r$ such that

$$\|h\|_{H^{1/2}(\gamma_0)} \le C_1' \|u\|_{H^1(\omega_1)} \le C_1' \|u\|_{H^1(\omega_r)}.$$

For the soft case, since $\partial u/\partial n = f$ and $\Delta u = 0$, then there exists a positive constant $C_1'$ independent of $r$ such that

$$\|f\|_{H^{-1/2}(\gamma_0)} \leq C_1' \|u\|_{H^1(\omega_1)} \leq C_1' \|u\|_{H^1(\omega_r)}.$$

The lower bounds then follow by taking $C_1 = 1/C_1'$. We now prove the upper bounds. First consider the hard case.

$$\int_{\omega_r} |\underset{\sim}{\operatorname{grad}}\, u|^2 = \int_{\partial\omega_r} u \frac{\partial u}{\partial n} = \int_{\gamma_0} u \frac{\partial u}{\partial n} = \int_{\gamma_0} h \frac{\partial u}{\partial n}.$$

Let $w \in H^1(\omega_1)$ be such that

$$\begin{aligned} \Delta w &= 0 && \text{in } \omega_1, \\ \frac{\partial w}{\partial n} &= 0 && \text{on } \gamma_1^+ \cup \gamma_1^-, \\ w &= h && \text{on } \gamma_0, \\ w &= 0 && \text{on } \gamma_1 \end{aligned}$$

It follows from a standard regularity result that

$$\|w\|_{H^1(\omega_1)} \leq C_2 \|h\|_{H^{1/2}(\gamma_0)}.$$

Thus, we have the following bound:

$$\begin{aligned} \|\underset{\sim}{\operatorname{grad}}\, u\|^2_{L^2(\omega_r)} &= \int_{\gamma_0} h \frac{\partial u}{\partial n} = \int_{\gamma_0} w \frac{\partial u}{\partial n} = \int_{\partial\omega_1} w \frac{\partial u}{\partial n} = \int_{\omega_1} \underset{\sim}{\operatorname{grad}}\, w \cdot \underset{\sim}{\operatorname{grad}}\, u \\ &\leq \|w\|_{H^1(\omega_1)} \|\underset{\sim}{\operatorname{grad}}\, u\|_{L^2(\omega_1)} \leq C_2 \|h\|_{H^{1/2}(\gamma_0)} \|\underset{\sim}{\operatorname{grad}}\, u\|_{L^2(\omega_r)}. \end{aligned}$$

It follows that

$$\|\underset{\sim}{\operatorname{grad}}\, u\|_{L^2(\omega_r)} \leq C_2 \|h\|_{H^{1/2}(\gamma_0)}. \tag{3.5.14}$$

Similar arguments hold for the soft case:

$$\int_{\omega_r} |\underset{\sim}{\operatorname{grad}}\, u|^2 = \int_{\partial\omega_r} \frac{\partial u}{\partial n} u = \int_{\gamma_0} \frac{\partial u}{\partial n} u = \int_{\gamma_0} g u.$$

Let $w \in H^1(\omega_1)$ be such that

$$\begin{aligned} \Delta w &= 0 && \text{in } \omega_1, \\ \frac{\partial w}{\partial n} &= 0 && \text{on } \gamma_1^+ \cup \gamma_1^- \cup \gamma_1, \\ \frac{\partial w}{\partial n} &= g && \text{on } \gamma_0, \\ \int_{\omega_1} w &= 0 \end{aligned}$$

It follows from a standard regularity result that

$$\|w\|_{H^1(\omega_1)} \le C_2 \|g\|_{-1/2,\gamma_0}.$$

Thus, we have the following bound:

$$\begin{aligned} \|\underset{\sim}{\operatorname{grad}}\, u\|^2_{L^2(\omega_r)} &= \int_{\gamma_0} u \frac{\partial u}{\partial n} = \int_{\gamma_0} u \frac{\partial w}{\partial n} = \int_{\partial\omega_1} u \frac{\partial w}{\partial n} = \int_{\omega_1} \underset{\sim}{\operatorname{grad}}\, u \cdot \underset{\sim}{\operatorname{grad}}\, w \\ &\le \|\underset{\sim}{\operatorname{grad}}\, u\|_{L^2(\omega_1)} \|w\|_{H^1(\omega_1)} \le C_2 \|\underset{\sim}{\operatorname{grad}}\, u\|_{L^2(\omega_r)} \|g\|_{H^{-1/2}(\gamma_0)}. \end{aligned}$$

It follows that

$$\|\underset{\sim}{\operatorname{grad}}\, u\|_{L^2(\omega_r)} \le C_2 \|g\|_{H^{-1/2}(\gamma_0)}. \tag{3.5.15}$$

From Lemma 3.5.2, $\int_{\gamma_s} u = 0$ for all $s \in [0, r]$, thus

$$\|u\|^2_{L^2(\omega_r)} = \int_0^r \int_{\gamma_s} |u|^2 \le C_2 \int_0^r \int_{\gamma_s} |\underset{\sim}{\operatorname{grad}}\, u|^2. \tag{3.5.16}$$

The upper bounds then follows from (3.5.14), (3.5.15) and (3.5.16). $\square$

**Theorem 3.5.9.** *Let $h$, $\underset{\approx}{\sigma}$, and $\underset{\sim}{u}$ satisfy* (3.5.7)–(3.5.11). *Then there exist constants $C_1$ and $C_2$ independent of $r \ge 1$ such that*

$$\begin{aligned} C_1 \|h\|_{H^{1/2}(\gamma_0)} &\le \|\underset{\approx}{\sigma}\|_{L^2(\omega_r)} \le C_2 \|h\|_{H^{1/2}(\gamma_0)}, \\ C_1 \|h\|_{H^{1/2}(\gamma_0)} &\le \|\underset{\approx}{\varepsilon}(\underset{\sim}{u})\|_{L^2(\omega_r)} \le C_2 \|h\|_{H^{1/2}(\gamma_0)}. \end{aligned}$$

*Proof.* We first prove the lower bounds. Since $u_2 = h$ on $\gamma_0$, by the trace theorem and Korn's inequality, there exists a constant $C_1'$ independent $r$ such that

$$\begin{aligned}\|h\|_{H^{1/2}(\gamma_0)} &\le C_1'\|u_2\|_{H^1(\omega_1)} \le C_1' \min(\|\underset{\approx}{\varepsilon}(\underset{\sim}{u})\|_{L^2(\omega_1)}, \|\underset{\approx}{\sigma}\|_{L^2(\omega_1)}) \\ &\le C_1' \min(\|\underset{\approx}{\varepsilon}(\underset{\sim}{u})\|_{L^2(\omega_r)}, \|\underset{\approx}{\sigma}\|_{L^2(\omega_r)}).\end{aligned}$$

The lower bounds then follow by taking $C_1 = 1/C_1'$. Now we prove the upper bounds. By (3.5.8)–(3.5.11), $\underset{\sim}{\operatorname{div}}\underset{\approx}{\sigma} = \underset{\sim}{0}$ in $\omega_r$, $\underset{\approx}{\sigma}\underset{\sim}{n} = \underset{\sim}{0}$ on $\gamma_r^+ \cup \gamma_r^-$, $\underset{\sim}{n}^T\underset{\approx}{\sigma}\underset{\sim}{n} = 0$ on $\gamma_0$, and $\int_{\gamma_r}(\underset{\approx}{\sigma}\underset{\sim}{n}) \cdot \underset{\sim}{u} = 0$. Then we have

$$\|\underset{\approx}{\sigma}\|^2_{L^2(\omega_r)} \le C_2 \int_{\omega_r} \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{u}) = C_2 \int_{\gamma_0} \underset{\sim}{s}^t \underset{\approx}{\sigma}\underset{\sim}{n} h. \tag{3.5.17}$$

Let $\underset{\sim}{w}$ be such that

$$\begin{aligned}
\underset{\approx}{\tau} &= A\underset{\approx}{\varepsilon}(\underset{\sim}{w}) && \text{in } \omega_1, \\
\underset{\sim}{\operatorname{div}}\underset{\approx}{\tau} &= \underset{\sim}{0} && \text{in } \omega_1, \\
\underset{\approx}{\tau}\underset{\sim}{n} &= \underset{\sim}{0} && \text{on } \gamma_1^+ \cup \gamma_1^-, \\
\underset{\sim}{n}^T \underset{\approx}{\tau}\underset{\sim}{n} &= 0 && \text{on } \gamma_0, \\
\underset{\sim}{w} \cdot \underset{\sim}{s} &= h && \text{on } \gamma_0, \\
\underset{\sim}{w} &= \underset{\sim}{0} && \text{on } \gamma_1.
\end{aligned} \tag{3.5.18}$$

Then

$$\|\underset{\sim}{w}\|_{H^1(\omega_1)} \le C_2 \|h\|_{H^{1/2}(\gamma_0)}. \tag{3.5.19}$$

Then by (3.5.17), (3.5.10), (3.5.11), (3.5.18) and (3.5.19), we obtain the following bound:

$$\begin{aligned}\|\underset{\approx}{\sigma}\|^2_{L^2(\omega_r)} &\le C_2 \int_{\gamma_0} \underset{\sim}{s}^T \underset{\approx}{\sigma}\underset{\sim}{n} h = C_2 \int_{\gamma_0} \underset{\sim}{s}^T (\underset{\approx}{\sigma}\underset{\sim}{n})(\underset{\sim}{w} \cdot \underset{\sim}{s}) = C_2 \int_{\partial\omega_1} (\underset{\approx}{\sigma}\underset{\sim}{n}) \cdot \underset{\sim}{w} \\ &= C_2 \int_{\omega_1} \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{w}) \le C_2 \|\underset{\approx}{\sigma}\|_{L^2(\omega_1)} \|\underset{\sim}{w}\|_{H^1(\omega_1)} \le C_2 \|\underset{\approx}{\sigma}\|_{L^2(\omega_r)} \|h\|_{H^{1/2}(\gamma_0)}\end{aligned}$$

The upper bounds then follow. $\square$

**Lemma 3.5.10.** *Let* $\underset{\sim}{u} \in \underset{\sim}{V}_h$ *be the unique solution to* (3.5.12). *If* $h$ *is an even function in* $x_3$, *then* $u_1$ *is an odd function in* $x_3$ *and* $u_2$ *is an even function in* $x_3$.

*Proof.* Define $\bar{u}_1(x,y) = -u_1(x,-y)$ and $\bar{u}_2(x,y) = u_2(x,-y)$. It is easy to check that $(\bar{u}_1, \bar{u}_2)$ is also a solution to (3.5.12). By uniqueness of the solution, $\bar{u}_1 = u_1$ and $\bar{u}_2 = u_2$. Hence $u_1$ is an odd function in $y$ and $u_2$ is an even function in $y$. □

## Sec 3.6. Order Estimation for the Scaled Terms

In this section, we estimate the orders of the solutions to (3.4.1) and (3.4.3) by using the results of Section 3.5.

**Estimates for $\phi$.** Consider the solution $\phi$ to (3.4.1) and define $u$ on $\omega_r$, where $r = L/t$, by

$$u(x, x_3) = \phi(tx, x_3). \tag{3.6.1}$$

The $u$ satisfies (3.5.1)–(3.5.4) with

$$h(0, x_3) = h_s(0, x_3), \quad g(0, x_3) = 2t(1+\nu)/Ef(0, x_3).$$

The following result therefore follows from Theorem 3.5.8.

**Theorem 3.6.1.** *Let* $L > 0$, $0 < t \le 1$, $h_s \in H^{1/2}(\gamma_0)$ *or* $f \in H^{-1/2}(\gamma_0)$ *given, and* $\phi$ *the solution to* (3.4.1). *Then there exist constants* $C_1$, $C_2$ *independent of* $t$ *such that*

$$\begin{aligned} C_1 t^{3/2} \|f\|_{H^{-1/2}(\gamma_0)} &\le \|\phi\|_{L^2(\omega)} + \left\| \frac{\partial \phi}{\partial x_3} \right\|_{L^2(\omega)} + t \left\| \frac{\partial \phi}{\partial \xi} \right\|_{L^2(\omega)} \\ &\le C_2 t^{3/2} \|f\|_{H^{-1/2}(\gamma_0)} \quad \textit{soft case}, \\ C_1 t^{1/2} \|h_s\|_{H^{1/2}(\gamma_0)} &\le \|\phi\|_{L^2(\omega)} + \left\| \frac{\partial \phi}{\partial x_3} \right\|_{L^2(\omega)} + t \left\| \frac{\partial \phi}{\partial \xi} \right\|_{L^2(\omega)} \\ &\le C_2 t^{1/2} \|h_s\|_{H^{1/2}(\gamma_0)} \quad \textit{hard case}. \end{aligned}$$

**Theorem 3.6.2.** *Let $\phi$ be the solution to (3.4.1), or (3.4.1) and (3.4.2). For all $0 \le \alpha \le L$, let $\tilde{\omega}_\alpha = \omega \backslash \omega_\alpha$. Then there exist positive constants $\lambda$ and $C$ independent of $t$ such that*

$$\|\phi\|_{L^2(\tilde{\omega}_\alpha)} + \left\| \frac{\partial \phi}{\partial x_3} \right\|_{L^2(\tilde{\omega}_\alpha)} + t \left\| \frac{\partial \phi}{\partial \xi} \right\|_{L^2(\tilde{\omega}_\alpha)} \le C t^{3/2} e^{-\lambda\alpha/2t} \|f\|_{H^{-1/2}(\gamma_0)} \quad \textit{soft case},$$

$$\|\phi\|_{L^2(\tilde{\omega}_\alpha)} + \left\| \frac{\partial \phi}{\partial x_3} \right\|_{L^2(\tilde{\omega}_\alpha)} + t \left\| \frac{\partial \phi}{\partial \xi} \right\|_{L^2(\tilde{\omega}_\alpha)} \le C t^{1/2} e^{-\lambda\alpha/2t} \|h_s\|_{H^{1/2}(\gamma_0)} \quad \textit{hard case}.$$

*Proof.* We choose $r_1 = 0$, $r_2 = \alpha/t$, and $r = L/t$ in Theorem 3.5.4, to get

$$\|u\|^2_{H^1(\omega_{\alpha/t, L/t})} \le C e^{-\lambda\alpha/t} \int_{\omega_{L/t}} |\underset{\sim}{\operatorname{grad}}\, u|^2.$$

Making the change of variables in (3.6.1), this becomes

$$\|\phi\|^2_{L^2(\tilde{\omega}_\alpha)} + \left\| \frac{\partial \phi}{\partial x_3} \right\|^2_{L^2(\tilde{\omega}_\alpha)} + t^2 \left\| \frac{\partial \phi}{\partial \xi} \right\|^2_{L^2(\tilde{\omega}_\alpha)} \le C e^{-\lambda\alpha/t} \int_\omega \left( t^2 \left| \frac{\partial \phi}{\partial \xi} \right|^2 + \left| \frac{\partial \phi}{\partial x_3} \right|^2 \right).$$

The right hand side can be bounded by Lemma 3.6.1 and the result follows. □

Before deriving several consequences of the exponential decay properties, we prove three calculus lemmas.

**Lemma 3.6.3.** *Let $L > 0$, $f : [0, L] \to \mathbb{R}$. Suppose there exist $K > 0$, $\mu > 0$, and $\gamma \ge 0$ such that*

$$\left( \int_\alpha^L |f|^2 \right)^{1/2} \le K e^{-\alpha\mu} \qquad \textit{for all } \gamma \le \alpha \le L. \tag{3.6.2}$$

*Then*

$$\int_\alpha^L |f| \le \frac{e^2}{e-1} K \mu^{-1/2} e^{-\alpha\mu} \qquad \textit{for all } \gamma \le \alpha \le L, \tag{3.6.3}$$

$$\int_\alpha^L \left( \int_\xi^L |f| \right)^2 \le \frac{e^4}{2(e-1)^2} K^2 \mu^{-2} e^{-2\alpha\mu} \qquad \textit{for all } \gamma \le \alpha \le L, \tag{3.6.4}$$

$$\int_\alpha^L \xi^2 |f|^2 \le C K^2 \mu^{-2} e^{-2\alpha\mu} \qquad \textit{for all } \gamma \le \alpha \le L, \tag{3.6.5}$$

*where* $C = e^2 \sum_{i=0}^{\infty} (i + [\alpha\mu] + 1)^2 e^{-2i}$, $[\alpha\mu]$ *is the largest integer less than or equal to* $\alpha\mu$.

*Proof.* Fix any $\alpha \in [0, L]$, let $I_n = [n\mu^{-1}, (n+1)\mu^{-1}] \cap [\alpha, L]$. By (3.6.2), for all $\gamma \le \alpha \le L$,

$$\begin{aligned}
\int_\alpha^L |f| &= \sum_{n=[\alpha\mu]}^{[L\mu]} \int_{I_n} |f| \le \mu^{-1/2} \sum_{n=[\alpha\mu]}^{[L\mu]} \left( \int_{I_n} |f|^2 \right)^{1/2} \\
&\le \mu^{-1/2} \sum_{n=[\alpha\mu]}^{[L\mu]} \left( \int_{\max(n\mu^{-1},\gamma)}^{L} |f|^2 \right)^{1/2} \le K\mu^{-1/2} \sum_{n=[\alpha\mu]}^{[L\mu]} e^{-n} \\
&\le K\mu^{-1/2} \sum_{n=[\alpha\mu]}^{\infty} e^{-n} = K\mu^{-1/2} e^{-[\alpha\mu]} \sum_{n=0}^{\infty} e^{-n} \\
&\le \frac{e^2}{e-1} K\mu^{-1/2} e^{-\alpha\mu}.
\end{aligned}$$

This establish (3.6.3).

Let

$$g(\xi) = \int_\xi^L |f|.$$

By (3.6.3),

$$g(\xi) \le \frac{e^2}{e-1} K\mu^{-1/2} e^{-\xi\mu}, \quad \gamma \le \xi \le L.$$

Thus, for $\gamma \le \alpha \le L$,

$$\int_\alpha^L |g|^2 \, d\xi \le \int_\alpha^L \left( \frac{e^2}{e-1} \right)^2 K^2 \mu^{-1} e^{-2\xi\mu} \, d\xi \le \frac{1}{2} \mu^{-2} \left( \frac{e^2}{e-1} \right)^2 K^2 e^{-2\alpha\mu},$$

which is (3.6.4).

Finally, for $\gamma \le \alpha \le L$,

$$\begin{aligned}
\int_\alpha^L \xi^2 |f|^2 &= \sum_{n=[\alpha\mu]}^{[L\mu]} \int_{I_n} \xi^2 |f|^2 \le \mu^{-2} \sum_{n=[\alpha\mu]}^{[L\mu]} (n+1)^2 \int_{I_n} |f|^2 \\
&\le \mu^{-2} \sum_{n=[\alpha\mu]}^{[L\mu]} (n+1)^2 \int_{\max(n\mu^{-1},\gamma)}^{L} |f|^2 \le K^2 \mu^{-2} \sum_{n=[\alpha\mu]}^{[L\mu]} (n+1)^2 e^{-2n} \\
&= K^2 \mu^{-2} e^{-2[\alpha\mu]} \sum_{i=0}^{\infty} (i + [\alpha\mu] + 1)^2 e^{-2i} \le K^2 \mu^{-2} e^{-2\alpha\mu} e^2 \sum_{i=0}^{\infty} (i + [\alpha\mu] + 1)^2 e^{-2i}.
\end{aligned}$$

This proves (3.6.5).

**Lemma 3.6.4.** *Suppose that the hypotheses of Lemma 3.6.3 hold with $\gamma = t$ and $\mu = \lambda/(2t)$ for some positive $t$, $\lambda$, and that*

$$\left(\int_0^L |f|^2\right)^{1/2} \leq K.$$

*Then*

$$\int_0^L |f| \leq \left(\frac{e^{2-\lambda/2}}{e-1}\sqrt{\frac{2}{\lambda}} + 1\right) K t^{1/2}, \tag{3.6.6}$$

$$\int_0^L \left(\int_\xi^L |f|\right)^2 \leq \left[\frac{e^{4-\lambda}}{(e-1)^2}\frac{2}{\lambda^2} + \left(\frac{e^{2-\lambda/2}}{e-1}\sqrt{\frac{2}{\lambda}} + 1\right)^2\right] K^2 t^2, \tag{3.6.7}$$

$$\int_0^L \xi^2 |f|^2 \leq \left(\frac{4C}{\lambda^2 e^\lambda} + 1\right) K^2 t^2, \tag{3.6.8}$$

*where* $C = \sum_{i=0}^\infty e^2(i + [\lambda/2] + 1)^2 e^{-2i}$.

*Proof.* Taking $\alpha = t$ in (3.6.3) gives

$$\int_t^L |f| \leq \frac{e^2}{e-1} K \sqrt{\frac{2}{\lambda}} t^{1/2} e^{-\lambda/2}.$$

Moreover,

$$\int_0^t |f| \leq t^{1/2}\left(\int_0^t |f|^2\right)^{1/2} \leq K t^{1/2}.$$

The inequality (3.6.6) then follows.

By (3.6.4) with the same choice $\alpha = t$,

$$\int_t^L \left(\int_\xi^L |f|\right)^2 \leq \frac{e^4}{2(e-1)^2} K^2 \frac{4}{\lambda^2} t^2 e^{-\lambda}.$$

Moreover by (3.6.6),

$$\int_0^t \left(\int_\xi^L |f|\right)^2 \leq \int_0^t \left(\int_0^L |f|\right)^2 \leq \left(\frac{e^{2-2/\lambda}}{e-1}\sqrt{\frac{2}{\lambda}} + 1\right)^2 K^2 t^2.$$

The inequality (3.6.7) then follows.

Finally by (3.6.5),

$$\int_t^L \xi^2|f|^2 \le CK^2\frac{4}{\lambda^2}t^2e^{-\lambda},$$

where $C = \sum_{i=0}^{\infty} e^2(i+[\lambda/2]+1)^2e^{-2i}$. Moreover,

$$\int_0^t \xi^2|f|^2 \le t^2\int_0^t |f|^2 \le K^2t^2.$$

The inequality (3.6.8) then follows. □

**Lemma 3.6.5.** *Let $L>0$, $0<t\le L$, and $\sigma:\omega\to\mathbb{R}$ be given. Set $\tilde{\omega}_\alpha = \omega\backslash\omega_\alpha$. Suppose that there exist constants $\lambda$, $M>0$, and $s\in\mathbb{R}$ independent of $t$ such that*

$$\|\sigma\|_{L^2(\tilde{\omega}_\alpha)} \le Mt^se^{-\lambda\alpha/2t} \quad \textit{for all } t\le\alpha\le L, \textit{ and for } \alpha=0. \tag{3.6.9}$$

*Then there exists a constant $C$ independent of $t$ such that*

$$\|\sigma\|_{L^1(\tilde{\omega}_\alpha)} \le CMt^{s+1/2}e^{-\lambda\alpha/2t} \qquad \textit{for all } t\le\alpha\le L, \tag{3.6.10}$$

$$\left\|\int_\xi^L |\sigma|\,d\zeta\right\|_{L^2(\tilde{\omega}_\alpha)} \le CMt^{s+1}e^{-\lambda\alpha/2t} \qquad \textit{for all } t\le\alpha\le L, \tag{3.6.11}$$

$$\|\sigma\|_{L^1(\omega)} \le CMt^{s+1/2}, \tag{3.6.12}$$

$$\left\|\int_\xi^L |\sigma|\,d\zeta\right\|_{L^2(\omega)} \le CMt^{s+1}, \tag{3.6.13}$$

$$\|\xi\sigma\|_{L^2(\omega)} \le CMt^{s+1}. \tag{3.6.14}$$

*Proof.* We apply Lemmas 3.6.3 and 3.6.4. Let

$$f(\xi) = \left(\int_{-1/2}^{1/2} |\sigma(\xi,x_3)|^2\,dx_3\right)^{1/2}, \quad K = Mt^s, \quad \mu = \frac{\lambda}{2t}, \quad \gamma = t. \tag{3.6.15}$$

By (3.6.9), we obtain

$$\left(\int_\alpha^L |f|^2\right)^{1/2} = \|\sigma\|_{L^2(\tilde{\omega}_\alpha)} \le Mt^se^{-\lambda\alpha/2t} = Ke^{-\alpha\mu} \qquad \text{for all } \gamma\le\alpha\le L,$$

$$\left(\int_0^L |f|^2\right)^{1/2} = \|\sigma\|_{L^2(\omega)} \le K.$$

Thus, the conditions of Lemma 3.6.3 and 3.6.4 are satisfied. By (3.6.3), (3.6.15) and Cauchy's inequality, we obtain

$$\begin{aligned}\|\sigma\|_{L^1(\tilde\omega_\alpha)} &= \int_\alpha^L \int_{-1/2}^{1/2} |\sigma| \le \int_\alpha^L |f| \le CK\mu^{-1/2}e^{-\alpha\mu} \\ &\le CMt^{s+1/2}e^{-\lambda\alpha/2t} \qquad \text{for all } t \le \alpha \le L,\end{aligned}$$

which is (3.6.10).

Moreover, by (3.6.6) and Cauchy's inequality,

$$\|\sigma\|_{L^1(\omega)} \le \int_0^L |f| \le CKt^{1/2} \le CMt^{s+1/2},$$

so (3.6.12) holds.

To show (3.6.11), take any $v \in L^2(\tilde\omega_\alpha)$ with $\|v\|_{L^2(\tilde\omega_\alpha)} = 1$. By Cauchy's inequality and (3.6.4), we obtain

$$\begin{aligned}&\int_{-1/2}^{1/2}\int_\alpha^L \left(\int_\xi^L |\sigma(\zeta, x_3)|\, d\zeta\right) v(\xi, x_3)\, d\xi dx_3 \\ &= \int_\alpha^L \int_\xi^L \int_{-1/2}^{1/2} |\sigma(\zeta, x_3)| v(\xi, x_3)\, dx_3 d\zeta d\xi \\ &\le \int_\alpha^L \left[\int_\xi^L \left(\int_{-1/2}^{1/2} |\sigma(\zeta, x_3)|^2\, dx_3\right)^{1/2} d\zeta\right] \left(\int_{-1/2}^{1/2} |v|^2\, dx_3\right)^{1/2} d\xi \\ &\le \left[\int_\alpha^L \left(\int_\xi^L |f|\, d\zeta\right)^2 d\xi\right]^{1/2} \le CK\mu^{-1}e^{-\alpha\mu} \\ &= CMt^{s+1}e^{-\lambda\alpha/2t} \qquad \text{for all } t \le \alpha \le L.\end{aligned}$$

This shows (3.6.11).

Similarly, but using (3.6.7) instead of (3.6.4), we get

$$\begin{aligned}&\int_{-1/2}^{1/2}\int_0^L \left(\int_\xi^L |\sigma(\zeta, x_3)|\, d\zeta\right) v(\xi, x_3)\, d\xi dx_3 \le \left[\int_0^L \left(\int_\xi^L |f|\, d\zeta\right)^2\right]^{1/2} \\ &\le CKt \le CMt^{s+1},\end{aligned}$$

i.e., (3.6.13).

Finally, take any $v \in L^2(\omega)$ with $\|v\|_{L^2(\omega)} = 1$. By Cauchy's inequality and (3.6.8), we obtain

$$\int_0^L \int_{-1/2}^{1/2} \xi \sigma v = \int_0^L \xi \left( \int_{-1/2}^{1/2} |\sigma|^2 \right)^{1/2} \left( \int_{-1/2}^{1/2} |v|^2 \right)^{1/2} d\xi$$
$$\leq C \left( \int_0^L \xi^2 f^2 \right)^{1/2} \leq CKt = CMt^{s+1},$$

showing (3.6.14) and completing the lemma. □

**Theorem 3.6.6.** *Let $L > 0$, $0 < t \leq 1$, $h \in H^{1/2}(\gamma_0)$, $f \in H^{-1/2}(\gamma_0)$ given, and $\phi$ the solution to* (3.4.1)*. Then there exists a constant $C$ independent of $t$ such that*

*(1) In the soft case*

$$\|\xi\phi\|_{L^2(\omega)} + \left\| \xi \frac{\partial \phi}{\partial x_3} \right\|_{L^2(\omega)} + t \left\| \xi \frac{\partial \phi}{\partial \xi} \right\|_{L^2(\omega)} \leq Ct^{5/2} \|f\|_{H^{-1/2}(\gamma_0)}, \tag{3.6.16}$$

$$\|\phi\|_{H^{-1,0}(\omega)} + \left\| \frac{\partial \phi}{\partial x_3} \right\|_{H^{-1,0}(\omega)} + t \left\| \frac{\partial \phi}{\partial \xi} \right\|_{H^{-1,0}(\omega)} \leq Ct^{5/2} \|f\|_{H^{-1/2}(\gamma_0)}, \tag{3.6.17}$$

*(2) In the hard case*

$$\|\xi\phi\|_{L^2(\omega)} + \left\| \xi \frac{\partial \phi}{\partial x_3} \right\|_{L^2(\omega)} + t \left\| \xi \frac{\partial \phi}{\partial \xi} \right\|_{L^2(\omega)} \leq Ct^{3/2} \|h_s\|_{H^{1/2}(\gamma_0)}, \tag{3.6.18}$$

$$\|\phi\|_{H^{-1,0}(\omega)} + \left\| \frac{\partial \phi}{\partial x_3} \right\|_{H^{-1,0}(\omega)} + t \left\| \frac{\partial \phi}{\partial \xi} \right\|_{H^{-1,0}(\omega)} \leq Ct^{3/2} \|h_s\|_{H^{1/2}(\gamma_0)}. \tag{3.6.19}$$

*Proof.* For $0 \leq \alpha \leq L$, let $\tilde{\omega}_\alpha = \omega \backslash \omega_\alpha$. Consider first the hard case. By Theorem 3.6.2, there exist positive constants $\lambda$, $C_1$ independent of $t$ such that

$$\|\phi\|_{L^2(\tilde{\omega}_\alpha)} \leq C_1 t^{1/2} e^{-\lambda\alpha/2t} \|h_s\|_{H^{1/2}(\gamma_0)} \qquad \text{for all } 0 \leq \alpha \leq L.$$

Thus may take $\sigma = \phi$, $s = 1/2$, and $M = C_1 \|h_s\|_{H^{1/2}(\gamma_0)}$ in Lemma 3.6.5. Then the desired bound on the first term on the left hand side of (3.6.18) follows immediately from (3.6.14).

Next take any $w \in H^{1,0}(\omega)$ with $w(0,x_3) = w(L,x_3) = 0$ and $\|w\|_{H^{1,0}(\omega)} = 1$. We obtain

$$\int_{-1/2}^{1/2}\int_0^L \phi w \, d\xi dx_3 = \int_{-1/2}^{1/2}\int_0^L \left(\int_\xi^L \phi \, d\zeta\right) \frac{\partial w}{\partial \xi} \, d\xi dx_3 \leq \left\| \int_\xi^L \phi \right\|_{L^2(\omega)}.$$

The desired bound on the first term on the left hand side of (3.6.19) follows immediately from (3.6.11). We have thus bounded the first terms on the left hand sides of (3.6.18) and (3.6.19). Identical argumentation gives the same bounds for remaining terms, and the soft case may be treated in the same way. □

**Estimates for $\psi$ and $\mu$.**

Consider the solution $(\psi,\mu)$ to (3.4.3) and define $(u_1,u_2)$ on $\omega_r$, where $r = L/t$, by

$$\psi(\xi,x_3) = \frac{1}{t} u_1(t^{-1}\xi, x_3), \quad \mu(\xi,x_3) = -u_2(t^{-1}\xi,x_3). \tag{3.6.20}$$

Then $(u_1,u_2)$ satisfies (3.5.7)–(3.5.11) with $h(0,x_3) = h_3(0,x_3)$. The following result then follows from Theorem 3.5.9.

**Theorem 3.6.7.** *Let $L > 0$, $0 < t \leq 1$, and $h_3 \in H^{1/2}(\gamma_0)$ be given, and let $(\psi,\mu)$ be the solution to* (3.4.3). *Then there exist constants $C_1$ and $C_2$ independent of $t$ such that*

$$C_1 t^{1/2}\|h_3\|_{H^{1/2}(\gamma_0)} \leq t^2 \left\|\frac{\partial \psi}{\partial \xi}\right\|_{L^2(\omega)} + t\left\|\frac{\partial \psi}{\partial x_3} + \frac{\partial \mu}{\partial \xi}\right\|_{L^2(\omega)} + \left\|\frac{\partial \mu}{\partial x_3}\right\|_{L^2(\omega)}$$
$$\leq C_2 t^{1/2}\|h_3\|_{H^{1/2}(\gamma_0)}.$$

□

**Theorem 3.6.8.** *Let $L > 0$, $0 < t \le 1$, and $h_3 \in H^{1/2}(\gamma_0)$ be given, and let $(\psi, \mu)$ be the solution to (3.4.3). For $0 \le \alpha \le L$ let $\tilde{\omega}_\alpha = \omega \backslash \omega_\alpha$. Then there exist constants $\lambda$ and $C$ independent of $t$ such that*

$$t^2 \left\| \frac{\partial \psi}{\partial \xi} \right\|_{L^2(\tilde{\omega}_\alpha)} + t \left\| \frac{\partial \psi}{\partial x_3} + \frac{\partial \mu}{\partial \xi} \right\|_{L^2(\tilde{\omega}_\alpha)} + \left\| \frac{\partial \mu}{\partial x_3} \right\|_{L^2(\tilde{\omega}_\alpha)}$$
$$\le C t^{1/2} e^{-\lambda\alpha/2t} \|h_3\|_{H^{1/2}(\gamma_0)} \quad \text{for all } t \le \alpha \le L.$$

*Proof.* We choose $r_1 = 0$, $r_2 = \alpha/t$, and $r = L/t$ in Theorem 3.5.5, to get for all $t \le \alpha \le L$,

$$\int_{\omega_{\alpha/t, L/t}} |\underset{\approx}{\varepsilon}(\underset{\sim}{u})|^2 \le C e^{-\lambda\alpha/t} \int_{\omega_{L/t}} |\underset{\approx}{\varepsilon}(\underset{\sim}{u})|^2.$$

Making the change of variables in (3.6.20), this becomes

$$t^4 \left\| \frac{\partial \psi}{\partial \xi} \right\|^2_{L^2(\tilde{\omega}_\alpha)} + t^2 \left\| \frac{\partial \psi}{\partial x_3} + \frac{\partial \mu}{\partial \xi} \right\|^2_{L^2(\tilde{\omega}_\alpha)} + \left\| \frac{\partial \mu}{\partial x_3} \right\|^2_{L^2(\tilde{\omega}_\alpha)}$$
$$\le C e^{-\lambda\alpha/t} \left( t^4 \left\| \frac{\partial \psi}{\partial \xi} \right\|^2_{L^2(\omega)} + t^2 \left\| \frac{\partial \psi}{\partial x_3} + \frac{\partial \mu}{\partial \xi} \right\|_{L^2(\omega)} + \left\| \frac{\partial \mu}{\partial x_3} \right\|^2_{L^2(\omega)} \right).$$

The right hand side can be bounded by Theorem 3.6.7 and the result follows. □

**Theorem 3.6.9.** *Let $L > 0$, $0 < t \le 1$, and $h_3 \in H^{1/2}(\gamma_0)$ be given, and let $(\psi, \mu)$ be the solution to (3.4.3). Then there exists a constant $C$ independent of $t$ such that*

$$t^2 \left\| \xi \frac{\partial \psi}{\partial \xi} \right\|_{L^2(\omega)} + t \left\| \xi \left( \frac{\partial \psi}{\partial x_3} + \frac{\partial \mu}{\partial \xi} \right) \right\|_{L^2(\omega)} + \left\| \xi \frac{\partial \mu}{\partial x_3} \right\|_{L^2(\omega)} \le C t^{3/2} \|h_3\|_{H^{1/2}(\gamma_0)}, \tag{3.6.21}$$

$$t^2 \left\| \frac{\partial \psi}{\partial \xi} \right\|_{H^{-1,0}(\omega)} + t \left\| \frac{\partial \psi}{\partial x_3} + \frac{\partial \mu}{\partial \xi} \right\|_{H^{-1,0}(\omega)} + \left\| \frac{\partial \mu}{\partial x_3} \right\|_{H^{-1,0}(\omega)} \le C t^{3/2} \|h_3\|_{H^{1/2}(\gamma_0)}. \tag{3.6.22}$$

*Proof.* First, we bound the first term on the left hand side of (3.6.21). By Lemma 3.6.8 and Theorem 3.6.7, there exist positive constants $\lambda$ and $C_1$ independent of $t$ such that

$$\left\| \frac{\partial \psi}{\partial \xi} \right\|_{L^2(\tilde{\omega}_\alpha)} \le C_1 t^{-3/2} e^{-\lambda\alpha/2t} \|h_3\|_{H^{1/2}(\gamma_0)} \quad \text{for all } t \le \alpha \le L, \text{ and for } \alpha = 0. \tag{3.6.23}$$

Then we let $\sigma = \partial\psi/\partial\xi$, $s = -3/2$, and $M = C_1\|h_3\|_{H^{1/2}(\gamma_0)}$ in Lemma 3.6.5, and from (3.6.14) we obtain

$$\left\|\xi\frac{\partial\psi}{\partial\xi}\right\|_{L^2(\omega)} \le Ct^{-1/2}\|h_3\|_{H^{1/2}(\gamma_0)}$$

as desired.

Given any $w \in H^{1,0}(\omega)$ with $w(0,x_3) = w(L,x_3) = 0$ and $\|w\|_{H^{1,0}(\omega)} = 1$,

$$\int_{-1/2}^{1/2}\int_0^L \frac{\partial\psi}{\partial\xi} w\, d\xi dx_3 = \int_{-1/2}^{1/2}\int_0^L \left(\int_\xi^L \frac{\partial\psi}{\partial\zeta}\, d\zeta\right)\frac{\partial w}{\partial\xi}\, d\xi dx_3.$$

Then by (3.6.11),

$$\int_{-1/2}^{1/2}\int_0^L \left(\int_\xi^L \frac{\partial\psi}{\partial\zeta}\, d\zeta\right)\frac{\partial w}{\partial\xi}\, d\xi dx_3 \le C\left\|\int_\xi^L \frac{\partial\psi}{\partial\zeta}\, d\zeta\right\|_{L^2(\omega)} \le Ct^{-1/2}\|h_3\|_{H^{1/2}(\gamma_0)}.$$

The desired bound on the first term of (3.6.22) then follows. The bounds on the other terms of (3.6.21) and (3.6.22) can be shown in the same way. □

## Sec 3.7. $L^2$ Estimation

In the previous section we obtained bounds on certain derivatives of $\psi$ and $\mu$. In this section, we estimate the order of $\|\psi\|_{L^2(\omega)}$ and $\|\mu\|_{L^2(\omega)}$. The result will be used in the next section.

**Theorem 3.7.1.** *Let $h_3 \in H^{1/2}(\gamma_0)$ an even function of $x_3$ with $\int_{\gamma_0} h_3 = 0$, and let $(\psi, \mu)$ be the solution to (3.4.3). Then*

$$\|\psi\|_{L^2(\omega)} \le Ct^{-1/2}\|h_3\|_{H^{1/2}(\gamma_0)}, \quad \|\mu\|_{L^2(\omega)} \le C\|h_3\|_{H^{1/2}(\gamma_0)}.$$

*Proof.* By (3.4.3) $\psi = 0$ on $\gamma_L$. Thus

$$\psi = \int_L^\xi \frac{\partial\psi(\xi,x_3)}{\partial\xi}\, d\xi.$$

In view of (3.6.23) we may apply Lemma 3.6.5 to get

$$\|\psi\|_{L^2(\omega)} = \left\| \int_\xi^L \frac{\partial \psi(\zeta, x_3)}{\partial \zeta} \right\|_{L^2(\omega)} \le C t^{-1/2} \|h_3\|_{H^{1/2}(\gamma_0)}.$$

Next we prove the estimate for $\mu$. Let

$$\begin{aligned} \mu^{(1)}(\xi) &= \int_{-1/2}^{1/2} \mu(\xi, s)\, d\theta, \\ \mu^{(2)}(\xi, x_3) &= \mu(\xi, x_3) - \mu^{(1)}(\xi). \end{aligned} \tag{3.7.1}$$

By Theorem 3.6.7,

$$\left\| \frac{\partial \mu}{\partial x_3} \right\|_{L^2(\omega)} \le C t^{1/2} \|h_3\|_{H^{1/2}(\gamma_0)}.$$

Since $\int_{\gamma_\xi} \mu^{(2)} = 0$ for all $0 \le \xi \le L$, then

$$\|\mu^{(2)}\|_{L^2(\omega)} \le C \left\| \frac{\partial \mu^{(2)}}{\partial x_3} \right\|_{L^2(\omega)} = C \left\| \frac{\partial \mu}{\partial x_3} \right\|_{L^2(\omega)} \le C t^{1/2} \|h_3\|_{H^{1/2}(\gamma_0)}. \tag{3.7.2}$$

It remains to bound $\mu^{(1)}$ in $L^2(\omega)$. By Lemma 3.5.10 and (3.6.20), $\psi$ is an odd function in $x_3$, and so $\psi(\xi, 0) = 0$. Therefore, we have the identity

$$\psi(\xi, x_3) = \int_0^{x_3} \frac{\partial \psi}{\partial x_3}(\xi, s)\, d\theta = \int_0^{x_3} \left( \frac{\partial \psi}{\partial x_3} + \frac{\partial \mu}{\partial \xi} \right)(\xi, s)\, d\theta - \int_0^{x_3} \frac{\partial \mu}{\partial \xi}(\xi, s)\, d\theta. \tag{3.7.3}$$

Let

$$\psi^{(1)}(\xi, x_3) = \psi(\xi, x_3) + x_3 \frac{d\mu^{(1)}}{d\xi}(\xi). \tag{3.7.4}$$

Then by (3.7.3) and (3.7.1), we obtain

$$\psi^{(1)} = \int_0^{x_3} \left[ \left( \frac{\partial \mu}{\partial \xi} + \frac{\partial \psi}{\partial x_3} \right) - \frac{\partial \mu^{(2)}}{\partial \xi} \right]. \tag{3.7.5}$$

Recall the expression for $\boldsymbol{\rho}$ in (3.3.4) and the equations (3.3.5) and (3.3.6). In particular, for any fixed value $s$ of the arclength coordinate, we have

$$\begin{aligned}
&\frac{\partial \rho_{nn}}{\partial \xi} + \frac{\partial \rho_{n3}}{\partial x_3} = 0, && \text{in } \omega, \\
&\frac{\partial \rho_{n3}}{\partial \xi} + \frac{\partial \rho_{33}}{\partial x_3} = 0, && \text{in } \omega, \\
&\rho_{n3} = 0, \quad \rho_{33} = 0 && \text{on } \gamma^+ \cup \gamma^-, \\
&\rho_{nn} = 0, \quad \mu = h_3 && \text{on } \gamma_0, \\
&\mu = \text{constant}, \quad \psi = 0 && \text{on } \gamma_L, \\
&\int_{\Gamma_L} \rho_{n3} = 0.
\end{aligned} \tag{3.7.6}$$

By Lemma 3.5.6 and (3.6.20) and (3.3.4),

$$\int_{\Gamma_L} x_3 \rho_{nn} = 0.$$

Take any $v = v(\xi) \in H^2(0, L)$ with $v(0) = 0$. Then

$$\underset{\approx}{\varepsilon}\begin{pmatrix} -x_3 \dfrac{dv}{d\xi} \\ v \end{pmatrix} = \begin{pmatrix} -x_3 \dfrac{d^2 v}{d\xi^2} & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, by the boundary values of $\rho_{nn}$, $\rho_{n3}$, $\rho_{33}$ on $\partial\omega$, and the facts that $v(0) = 0$, $-x_3 \partial dv/\partial d\xi$ is linear, and $v$ is constant on $\Gamma_L$, we obtain

$$\begin{aligned}
0 = \int_\omega \underset{\sim}{\widehat{\operatorname{div}}} \begin{pmatrix} \rho_{nn} & \rho_{n3} \\ \rho_{n3} & \rho_{33} \end{pmatrix} \begin{pmatrix} -x_3 \dfrac{dv}{d\xi} \\ v \end{pmatrix} &= -\int_\omega \begin{pmatrix} \rho_{nn} & \rho_{n3} \\ \rho_{n3} & \rho_{33} \end{pmatrix} : \underset{\approx}{\varepsilon} \begin{pmatrix} -x_3 \dfrac{dv}{d\xi} \\ v \end{pmatrix} \\
+ \int_{\partial\omega} \left[ \begin{pmatrix} \rho_{nn} & \rho_{n3} \\ \rho_{n3} & \rho_{33} \end{pmatrix} \underset{\sim}{n} \right] \begin{pmatrix} -x_3 \dfrac{dv}{d\xi} \\ v \end{pmatrix} &= \int_\omega x_3 \rho_{nn} \frac{d^2 v}{d\xi^2},
\end{aligned}$$

where $\widehat{\operatorname{div}} f = \partial f/\partial \xi + \partial f/\partial x_3$. Since $\left\{ d^2 v/d\xi^2 \in H^2(0, L) \,|\, v(0) = 0 \right\} = L^2(0, L)$, it follows that

$$\int_{-1/2}^{1/2} x_3 \rho_{nn} = 0 \qquad \text{for all } \xi \in [0, L].$$

From the expression of $\rho_{nn}$ in (3.3.4) this implies

$$\int_{-1/2}^{1/2} x_3 \frac{\partial \psi}{\partial \xi} = -\frac{\nu}{t^2(1-\nu)} \int_{-1/2}^{1/2} x_3 \frac{\partial \mu}{\partial x_3}. \tag{3.7.7}$$

Now we differentiate both sides of (3.7.4) with respect to $\xi$, multiply by $x_3$ and integrate from $-1/2$ to $1/2$ in $x_3$ direction. Thus,

$$\begin{aligned} \frac{1}{12}\frac{d^2\mu^{(1)}}{d\xi^2} &= \int_{-1/2}^{1/2} x_3 \frac{\partial \psi^{(1)}}{\partial \xi}\, dx_3 - \int_{-1/2}^{1/2} x_3 \frac{\partial \psi}{\partial \xi}\, dx_3 \\ &= \int_{-1/2}^{1/2} \left[ x_3 \frac{\partial \psi^{(1)}}{\partial \xi} + \frac{\nu}{t^2(1-\nu)} x_3 \frac{\partial \mu}{\partial x_3} \right] dx_3, \end{aligned} \tag{3.7.8}$$

where we used (3.7.7) in the last step.

Now $\int_{\gamma_0} \mu = \int_{-1/2}^{1/2} h_3 = 0$. That is

$$\mu^{(1)}(0) = 0. \tag{3.7.9}$$

By (3.7.6) and the expression of $\rho_{n3}$ in (3.3.4), $\psi = 0$ on $\gamma_L$ and

$$\int_{\gamma_L} \frac{\partial \psi}{\partial x_3} + \frac{\partial \mu}{\partial \xi} = 0.$$

Thus

$$\int_{\gamma_L} \frac{\partial \mu}{\partial \xi} = 0.$$

That is

$$\frac{\partial \mu^{(1)}}{\partial \xi}(L) = 0. \tag{3.7.10}$$

From (3.7.8), (3.7.9), and (3.7.10) we can compute $\mu^{(1)}$:

$$\mu^{(1)}(\xi) = 12 \int_0^{\xi} \int_L^{\zeta} \int_{-1/2}^{1/2} x_3 \left[ \frac{\partial \psi^{(1)}}{\partial \xi}(\eta, x_3) + \frac{\nu}{t^2(1-\nu)} \frac{\partial \mu}{\partial x_3}(\eta, x_3) \right] dx_3\, d\eta\, d\zeta. \tag{3.7.11}$$

Now we consider the term $\left\| \int_0^{\xi} \int_L^{\zeta} \int_{-1/2}^{1/2} x_3 \partial\mu/\partial x_3 \right\|_{L^2(\omega)}$. By Theorem 3.6.7 and Lemma 3.6.9, there exist positive constants $\lambda$, $C$ independent of $t$ such that

$$\left\| \frac{\partial \mu}{\partial x_3} \right\|_{L^2(\tilde{\omega}_\alpha)} \le C t^{1/2} e^{-\lambda\alpha/2t} \|h_3\|_{H^{1/2}(\gamma_0)} \qquad \text{for all } t \le \alpha \le L \text{ and for } \alpha = 0.$$

Then by (3.6.11) and (3.6.13), we obtain

$$\left\| \int_{\zeta}^{L} \left| \frac{\partial \mu}{\partial x_3} \right| \right\|_{L^2(\tilde{\omega}_\alpha)} \le C t^{3/2} e^{-\lambda\alpha/2t} \|h_3\|_{H^{1/2}(\gamma_0)} \qquad \text{for all } t \le \alpha \le L \text{ and for } \alpha = 0.$$

Thus by (3.6.12) in Lemma 3.6.5, we obtain

$$\left\| \int_0^\xi \int_\zeta^L \int_{-1/2}^{1/2} x_3 \frac{\partial \mu}{\partial x_3} \right\|_{L^2(\omega)} \leq C \left\| \int_\zeta^L \left| \frac{\partial \mu}{\partial x_3} \right| \right\|_{L^1(\omega)} \leq Ct^2 \|h_3\|_{H^{1/2}(\gamma_0)}. \quad (3.7.12)$$

By (3.7.6) $\psi = 0$ on $\gamma_L$, and by (3.7.10) $d\mu^{(1)}/d\xi = 0$ on $\gamma_L$. Then by (3.7.4) we obtain

$$\psi^{(1)} = 0 \qquad \text{on } \gamma_L.$$

Thus by (3.7.5)

$$\begin{aligned} \int_0^\xi \int_L^\zeta \int_{-1/2}^{1/2} x_3 \frac{\partial \psi^{(1)}}{\partial \xi} &= \int_{-1/2}^{1/2} x_3 \int_0^\xi \psi^{(1)}(\zeta, x_3) \\ &= \int_{-1/2}^{1/2} x_3 \int_0^{x_3} \left[ \int_0^\xi \left( \frac{\partial \mu}{\partial \xi} + \frac{\partial \psi}{\partial x_3} \right) - \mu^{(2)}(\xi, x_3) + \mu^{(2)}(0, x_3) \right]. \end{aligned} \quad (3.7.13)$$

By (3.7.2),

$$\left\| \int_0^{x_3} \mu^{(2)} \right\|_{L^2(\omega)} \leq Ct^{1/2} \|h_3\|_{H^{1/2}(\gamma_0)}.$$

By (3.7.6) $\mu = h_3$ on $\gamma_0$. By (3.7.9) $\mu^{(1)} = 0$ on $\gamma_0$. Then $\mu^{(2)} = \mu - \mu^{(1)} = h_3$ on $\gamma_0$. Thus

$$\|\mu^{(2)}(0, x_3)\|_{L^2(\omega)} \leq C\|h_3\|_{L^2(\gamma_0)} \leq C\|h_3\|_{H^{1/2}(\gamma_0)}.$$

By Lemma 3.6.8 and 3.6.7, there exist positive constants $\lambda$, $C$ independent of $t$ such that

$$\left\| \frac{\partial \psi}{\partial x_3} + \frac{\partial \mu}{\partial \xi} \right\|_{L^2(\tilde{\omega}_\alpha)} \leq Ct^{-1/2} e^{-\lambda\alpha/2t} \|h_3\|_{H^{1/2}(\gamma_0)} \quad \text{for all } t \leq \alpha \leq L \text{ and for } \alpha = 0.$$

Then by (3.6.12) in Lemma 3.6.5,

$$\left\| \int_0^\xi \left( \frac{\partial \mu}{\partial \xi} + \frac{\partial \psi}{\partial x_3} \right) \right\|_{L^2(\omega)} \leq C\|h_3\|_{H^{1/2}(\gamma_0)}. \quad (3.7.14.)$$

By (3.7.13)–(3.7.14), we obtain

$$\left\| \int_0^\xi \int_L^\zeta \int_{-1/2}^{1/2} x_3 \frac{\partial \psi^{(1)}}{\partial \tau} \right\|_{L^2(\omega)} \leq C\|h_3\|_{H^{1/2}(\gamma_0)}. \quad (3.7.15)$$

By (3.7.11), (3.7.12), and (3.7.15), we obtain

$$\|\mu^{(1)}\|_{L^2(\omega)} \leq C\|h_3\|_{H^{1/2}(\gamma_0)}.$$

Together with (3.7.2), this gives the desired bound on $\mu$. $\square$

## Sec 3.8. Error Estimation

In this section, we estimate the orders of $\|\boldsymbol{\sigma}^t\|_{L^2(P)}$ and $\|\boldsymbol{\sigma}^t\|_{L^2(P_0)}$, where $\boldsymbol{\sigma}^t$ is the scaled boundary corrector in (3.2.8).

The order estimation depends on the results we have obtained about $\psi$, $\mu$ and $\phi$. By (3.3.2), (3.3.3) and (3.3.4), we recover $\boldsymbol{\rho}$ and $\vec{y}$ from $\psi$, $\mu$ and $\phi$. Let $\chi(\xi)$ be a cutoff function that assumes the value one in a neighborhood of the lateral boundary with width of $L/2$, and the value zero outside $Q$. Let $\bar{\boldsymbol{\rho}} = \boldsymbol{\rho}\chi$ and $\vec{\bar{y}} = \vec{y}\chi$. Then $(\bar{\boldsymbol{\rho}}, \vec{\bar{y}})$ is defined on the entire domain $P$. Decompose the three-dimensional boundary corrector $(\boldsymbol{\sigma}^t, \vec{u}^t)$ as $(\boldsymbol{\eta}, \vec{y}) + (\bar{\boldsymbol{\rho}}, \vec{\bar{y}})$. It follows directly from (3.2.8) and (3.2.9) that $(\boldsymbol{\eta}, \vec{z})$ is the solution to the following problem:

Find $(\boldsymbol{\eta}, \vec{z}) \in \boldsymbol{\Sigma}^t \times \boldsymbol{V}^t$ such that

$$\begin{aligned}
&a_0(\boldsymbol{\eta}, \boldsymbol{\tau}) + t^2 a_2(\boldsymbol{\eta}, \boldsymbol{\tau}) + t^4 a_4(\boldsymbol{\eta}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \vec{z}) \\
&\quad = -\left[a_0(\bar{\boldsymbol{\rho}}, \boldsymbol{\tau}) + t^2 a_2(\bar{\boldsymbol{\rho}}, \boldsymbol{\tau}) + t^4 a_4(\bar{\boldsymbol{\rho}}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \vec{\bar{y}})\right] \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma}^t, \\
&b(\boldsymbol{\eta}, \vec{v}) = \begin{cases} \int_\Gamma f^t(\vec{v}\cdot\vec{s}) - b(\bar{\boldsymbol{\rho}}, \vec{v}) & \text{soft case} \\ -b(\bar{\boldsymbol{\rho}}, \vec{v}) & \text{hard case} \end{cases} \quad \text{for all } \vec{v} \in \boldsymbol{V}^t.
\end{aligned} \tag{3.8.1}$$

For any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}^t$, $J\boldsymbol{\tau}|_Q \in \boldsymbol{\Sigma}_Q$, where $J = 1 - \xi/R$ is the Jacobi determinant in (3.3.1). For any $\vec{v} \in \boldsymbol{V}^t$, $\vec{\bar{v}} = \chi\vec{v} \in \boldsymbol{V}_Q$. Moreover $\chi\vec{v} = \vec{v}$ on $\Gamma$. Thus, from (3.3.8) and (3.3.9),

$$\begin{aligned}
&A_0^Q(\boldsymbol{\rho}, J\boldsymbol{\tau}) + t^2 A_2^Q(\boldsymbol{\rho}, J\boldsymbol{\tau}) + t^4 A_4^Q(\boldsymbol{\rho}, J\boldsymbol{\tau}) + B^Q(J\boldsymbol{\tau}, \vec{y}) = 0 \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \\
&B^Q(\boldsymbol{\rho}, \chi\vec{v}) = \begin{cases} \int_\Gamma f^t(\vec{v}\cdot\vec{s}) & \text{soft case} \\ 0 & \text{hard case} \end{cases} \quad \text{for all } \vec{v} \in \boldsymbol{V}_\beta.
\end{aligned} \tag{3.8.2}$$

By (3.8.1) and (3.8.2), we obtain

$$
\begin{aligned}
&a_0(\boldsymbol{\eta},\boldsymbol{\tau}) + t^2 a_2(\boldsymbol{\eta},\boldsymbol{\tau}) + t^4 a_4(\boldsymbol{\eta},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\vec{\omega}) \\
&\quad = -\left[a_0(\bar{\boldsymbol{\rho}},\boldsymbol{\tau}) + t^2 a_2(\bar{\boldsymbol{\rho}},\boldsymbol{\tau}) + t^4 a_4(\bar{\boldsymbol{\rho}},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\bar{\vec{y}})\right] \\
&\quad + \left[A_0^Q(\boldsymbol{\rho},J\boldsymbol{\tau}) + t^2 A_2^Q(\boldsymbol{\rho},J\boldsymbol{\tau}) + t^4 A_4^Q(\boldsymbol{\rho},J\boldsymbol{\tau}) + B^Q(J\boldsymbol{\tau},\vec{y})\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \boldsymbol{\tau}\in\boldsymbol{\Sigma}^t, \\
&b(\boldsymbol{\eta},\vec{v}) = -b(\bar{\boldsymbol{\rho}},\vec{v}) + B^Q(\boldsymbol{\rho},\bar{\vec{v}}) \qquad\qquad \text{for all } \vec{v}\in\boldsymbol{V}^t.
\end{aligned}
\tag{3.8.3}
$$

We shall obtain the necessary bounds on $(\boldsymbol{\eta},\vec{z})$ from (3.8.3).

**Expressions of error terms.** Since $\bar{\boldsymbol{\rho}}$ and $\bar{\vec{y}}$ vanish outside the domain $Q$, then the right hand side expressions of (3.8.3) are all integration expressions over the region $Q$. Recall that $\int_Q f = \int_{-1/2}^{1/2}\int_0^S\int_0^L fJ$ for any function $f$. We can write

$$
\begin{aligned}
&a_0(\boldsymbol{\eta},\boldsymbol{\tau}) + t^2 a_2(\boldsymbol{\eta},\boldsymbol{\tau}) + t^4 a_4(\boldsymbol{\eta},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\vec{\omega}) = -\int_Q (\boldsymbol{L}^{(1)} + \boldsymbol{L}^{(2)} + \boldsymbol{L}^{(3)}) : \boldsymbol{\tau}, \\
&b(\boldsymbol{\eta},\vec{v}) = -\int_Q \boldsymbol{M} : \boldsymbol{\varepsilon}(\vec{v}) - \langle \boldsymbol{N}^{(1)},\vec{v}\rangle - \langle \boldsymbol{N}^{(2)},\vec{v}\rangle - \langle \boldsymbol{N}^{(3)},\vec{v}\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
\int_Q \boldsymbol{L}^{(1)} : \boldsymbol{\tau} &= a_0(\bar{\boldsymbol{\rho}},\boldsymbol{\tau}) + t^2 a_2(\bar{\boldsymbol{\rho}},\boldsymbol{\tau}) + t^4 a_4(\bar{\boldsymbol{\rho}},\boldsymbol{\tau}) \\
&\quad - \left[A_0^Q(\boldsymbol{\rho},J\boldsymbol{\tau}) + t^2 A_2^Q(\boldsymbol{\rho},J\boldsymbol{\tau}) + t^4 A_4^Q(\boldsymbol{\rho},J\boldsymbol{\tau})\right],
\end{aligned}
\tag{3.8.4}
$$

$$
\int_Q \boldsymbol{L}^{(2)} : \boldsymbol{\tau} = b(\boldsymbol{\tau},\bar{\vec{y}}) - \int_Q \boldsymbol{\tau} : \boldsymbol{\varepsilon}(\vec{y}), \tag{3.8.5}
$$

$$
\int_Q \boldsymbol{L}^{(3)} : \boldsymbol{\tau} = \int_Q \boldsymbol{\tau} : \boldsymbol{\varepsilon}(\vec{y}) - B^Q(J\boldsymbol{\tau},\vec{y}), \tag{3.8.6}
$$

$$
\int_Q \boldsymbol{M} : \boldsymbol{\varepsilon}(\vec{v}) = b(\bar{\boldsymbol{\rho}},\vec{v}) - \int_Q \boldsymbol{\rho} : \boldsymbol{\varepsilon}(\vec{v}), \tag{3.8.7}
$$

$$
\langle \boldsymbol{N}^{(1)},\vec{v}\rangle = \int_Q \boldsymbol{\rho} : \boldsymbol{\varepsilon}(\vec{v}) - B^Q(J\boldsymbol{\rho},\vec{v}), \tag{3.8.8}
$$

$$
\langle \boldsymbol{N}^{(2)},\vec{v}\rangle = B^Q(J\boldsymbol{\rho},\vec{v}) - B^Q(\boldsymbol{\rho},\vec{v}), \tag{3.8.9}
$$

$$
\langle \boldsymbol{N}^{(3)},\vec{v}\rangle = B^Q(\boldsymbol{\rho},\vec{v}) - B^Q(\boldsymbol{\rho},\bar{\vec{v}}). \tag{3.8.10}
$$

By (3.3.11),

$$a_i(\bar{\boldsymbol{\rho}}, \boldsymbol{\tau}) = a_i(\chi_Q \bar{\boldsymbol{\rho}}, \boldsymbol{\tau}) = A_i^Q(\bar{\boldsymbol{\rho}}, J\boldsymbol{\tau}).$$

Thus by (3.8.4)

$$\int_Q \boldsymbol{L}^{(1)} : \boldsymbol{\tau} = A_0^\beta(\bar{\boldsymbol{\rho}} - \boldsymbol{\rho}, J\boldsymbol{\tau}) + t^2 A_2^\beta(\bar{\boldsymbol{\rho}} - \boldsymbol{\rho}, J\boldsymbol{\tau}) + t^4 A_4^\beta(\bar{\boldsymbol{\rho}} - \boldsymbol{\rho}, J\boldsymbol{\tau}). \tag{3.8.11}$$

From (3.8.5), we obtain

$$\int_Q \boldsymbol{L}^{(2)} : \boldsymbol{\tau} = \int_Q \boldsymbol{\tau} : \boldsymbol{\varepsilon}(\vec{\bar{y}} - \vec{y}).$$

Thus

$$\boldsymbol{L}^{(2)} = \boldsymbol{\varepsilon}(\vec{\bar{y}} - \vec{y}). \tag{3.8.12}$$

From (3.8.6) and (3.3.12), we obtain

$$\begin{aligned} \boldsymbol{L}^{(3)} = \frac{1}{1-\xi/R} \left[ \frac{1}{2} \left( \frac{\partial \psi}{\partial \theta} + \frac{1}{R}\phi \right) (\vec{n}\vec{s}^T + \vec{s}\vec{n}^T) \right. \\ \left. + \left( \frac{\partial \phi}{\partial \theta} - \frac{1}{R}\psi \right) \vec{s}\vec{s}^T + \frac{1}{2}\frac{\partial \mu}{\partial \theta} (\vec{e}_3 \vec{s}^T + \vec{s}\vec{e}_3^T) \right]. \end{aligned} \tag{3.8.13}$$

From (3.8.7), we obtain

$$\int_Q \boldsymbol{M} : \boldsymbol{\varepsilon}(\vec{v}) = \int_Q (\bar{\boldsymbol{\rho}} - \boldsymbol{\rho}) : \boldsymbol{\varepsilon}(\vec{v}).$$

Thus

$$\boldsymbol{M} = \bar{\boldsymbol{\rho}} - \boldsymbol{\rho}. \tag{3.8.14}$$

From (3.8.8) and (3.3.12), we obtain

$$\langle \boldsymbol{N}^{(1)}, \vec{v} \rangle = \int_{-1/2}^{1/2} \int_0^S \int_0^L \left[ \rho_{ns} \left( \frac{\partial v_n}{\partial \theta} + \frac{1}{R} v_s \right) + \rho_{ss} \left( \frac{\partial v_s}{\partial s} - \frac{1}{R} \right) + \rho_{3s} \frac{\partial v_s}{\partial \theta} \right]. \tag{3.8.15}$$

From (3.8.9) and the definition of $B^\beta$ in (3.3.10), we obtain

$$\begin{aligned} \langle \boldsymbol{N}^{(2)}, \vec{v} \rangle = B^Q(J\boldsymbol{\rho}, \vec{v}) - B^Q(\boldsymbol{\rho}, \vec{v}) = - \int_{-1/2}^{1/2} \int_0^S \int_0^L \left( \rho_{nn} \frac{\partial v_n}{\partial \xi} \right. \\ \left. + \rho_{ns} \frac{\partial v_s}{\partial \xi} + \rho_{n3} \frac{\partial v_3}{\partial \xi} + \rho_{n3} \frac{\partial v_n}{\partial x_3} + \rho_{s3} \frac{\partial v_s}{\partial x_3} + \rho_{33} \frac{\partial v_3}{\partial x_3} \right) \frac{\xi}{R} \, d\xi d\theta dx_3. \end{aligned} \tag{3.8.16}$$

From (3.8.10) and (3.3.10),

$$\langle \boldsymbol{N}^{(3)}, \vec{v} \rangle = B^Q((1-\chi)\boldsymbol{\rho}, \vec{v}) - \int_{-1/2}^{1/2} \int_0^S \int_0^L \frac{d\chi}{d\xi} (\rho_{nn} v_n + \rho_{ns} v_s + \rho_{n3} v_3) \, d\xi d\theta dx_3. \tag{3.8.17}$$

**Order estimation for $\boldsymbol{\eta}$ and $\vec{z}$.** We define a $t$-dependent norm on $\boldsymbol{\Sigma}^t$ by

$$|||\boldsymbol{\tau}|||_t = \left( \sum_{\alpha,\beta} \|\tau_{\alpha\beta}\|^2_{L^2(P)} + t^2 \sum_{\alpha} \|\tau_{\alpha 3}\|^2_{L^2(P)} + t^4 \|\tau_{33}\|^2_{L^2(P)} \right)^{1/2}.$$

The dual norm is given by

$$|||\boldsymbol{\eta}|||'_t = \left( \sum_{\alpha,\beta} \|\eta_{\alpha\beta}\|^2_{L^2(P)} + \frac{1}{t^2} \sum_{\alpha} \|\eta_{\alpha 3}\|^2_{L^2(P)} + \frac{1}{t^4} \|\eta_{33}\|^2_{L^2(P)} \right)^{1/2}.$$

On $\boldsymbol{V}^t$ define

$$\|\vec{v}\|_t = |||\boldsymbol{\varepsilon}(\vec{v})|||'_t.$$

Denote the dual norm as $\|\cdot\|'_t$. By Korn's inequality, $\|\cdot\|_t$ defines a norm on $\boldsymbol{V}^t$. We also define $|||\boldsymbol{\tau}|||_{t,P_0}$ similarly by restricting the domain to $P_0$. Note that

$$\int_P \boldsymbol{R} : \boldsymbol{\varepsilon}(\vec{v}) \le |||\boldsymbol{R}|||_t \|\vec{v}\|_t.$$

**Theorem 3.8.1.** *Let $F \in (\boldsymbol{\Sigma}^t, |||\cdot|||_t)'$ and $G \in (\boldsymbol{V}^t, \|\cdot\|_t)'$ be given, and let $(\boldsymbol{\sigma}, \vec{u}) \in \boldsymbol{\Sigma}^t \times \boldsymbol{V}^t$ the solution to the following problem:*

$$\begin{aligned} &a_0(\boldsymbol{\eta},\boldsymbol{\tau}) + t^2 a_2(\boldsymbol{\eta},\boldsymbol{\tau}) + t^4 a_4(\boldsymbol{\eta},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\vec{u}) = \langle F, \boldsymbol{\tau}\rangle \quad && \text{for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma}^t, \\ &b(\boldsymbol{\eta},\vec{v}) = \langle G, \vec{v}\rangle && \text{for all } \vec{v} \in \boldsymbol{V}^t. \end{aligned}$$

*Here $a_i$ and $b$ are defined in* (3.2.8). *Then there exists a constant $C$ independent of $t$ such that*

$$|||\boldsymbol{\eta}|||_t + \|\vec{u}\|_t \le C\left(|||F|||'_t + \|G\|'_t\right).$$

*where $a_i$ and $b$ are defined in* (3.2.8).

*Proof.* Let

$$A(\boldsymbol{\sigma},\boldsymbol{\tau}) = a_0(\boldsymbol{\sigma},\boldsymbol{\tau}) + t^2 a_2(\boldsymbol{\sigma},\boldsymbol{\tau}) + t^4 a_4(\boldsymbol{\sigma},\boldsymbol{\tau}) \quad \text{for all } \boldsymbol{\sigma},\boldsymbol{\tau} \in \boldsymbol{\Sigma}^t.$$

Note that $A$ is symmetric. and

$$A(\boldsymbol{\tau}, \boldsymbol{\tau}) = |||\boldsymbol{\tau}|||_t^2 \qquad \text{for all } \boldsymbol{\tau} \in \boldsymbol{\Sigma}^t.$$

Furthermore, for any $\vec{v} \in \boldsymbol{V}^t$, let $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}^t$ be defined by

$$\sigma_{\alpha\lambda} = \varepsilon_{\alpha\lambda}(\vec{v}), \qquad \sigma_{\alpha 3} = \frac{1}{t^2}\varepsilon_{\alpha 3}(\vec{v}), \qquad \sigma_{33} = \frac{1}{t^4}\varepsilon_{33}(\vec{v}).$$

Then

$$b(\boldsymbol{\sigma}, \vec{v}) = |||\boldsymbol{\sigma}|||_t \|\vec{v}\|_t.$$

The theorem then follows from Brezzi's theorem [8]. □

The following corollary is immediate.

**Corollary 3.8.2.** *Let* $\boldsymbol{L}^{(1)}$, $\boldsymbol{L}^{(2)}$, $\boldsymbol{L}^{(3)}$, $\boldsymbol{M}$, $\boldsymbol{N}^{(1)}$, $\boldsymbol{N}^{(2)}$, *and* $\boldsymbol{N}^{(3)}$ *be as in* (3.8.4)–(3.8.10), *and let* $\boldsymbol{\eta}$, $\vec{z}$ *satisfy* (3.8.1). *Then there is a constant* $C$ *independent of* $t$ *such that*

$$|||\boldsymbol{\eta}|||_t + \|\vec{z}\|_t \leq C\left(\sum_i |||L^{(i)}|||_t' + |||M|||_t + \sum_i \|N^{(i)}\|_t'\right). \tag{3.8.18}$$

We first show that some of the terms on the right hand side of (3.8.18) have negative exponential orders.

**Lemma 3.8.3.** *Let* $P_0$ *be an interior domain of* $P$ *(i.e.* $P_0 = \Omega_0 \times (-1/2, 1/2)$ *with* $\bar{\Omega}_0 \subset \Omega$*), and let* $\boldsymbol{\rho}$ *be defined in* (3.3.4). *Then there exist positive constants* $C_1$ *and* $C_2$ *such that*

$$\|\boldsymbol{\rho}\|_{L^2(P_0 \cap Q)} \leq \begin{cases} C_1 e^{-C_2/t}(\|f\|_{H^{-1/2}(\gamma_0)} + \|h_3\|_{H^{1/2}(\gamma_0)}) & \textit{soft case,} \\ C_1 e^{-C_2/t}(\|h_s\|_{H^{1/2}(\gamma_0)} + \|h_3\|_{H^{1/2}(\gamma_0)}) & \textit{hard case.} \end{cases}$$

*Proof.* By the expression of $\boldsymbol{\rho}$ in (3.3.4), the lemma follows from Theorems 3.6.2 and 3.6.8. □

**Corollary 3.8.4.** *Let* $\boldsymbol{L}^{(1)}$, $\boldsymbol{M}$, *and* $\boldsymbol{N}^{(3)}$ *be as in* (3.8.4), (3.8.7) *and* (3.8.10) *respectively. Then there exists positive constants* $C_1$, $C_2$ *such that*

$$|||\boldsymbol{L}^{(1)}|||_t' + \|\boldsymbol{M}\|_t + \|\boldsymbol{N}^{(3)}\|_t' \\ \leq \begin{cases} C_1 e^{-C_2/t}(\|f\|_{H^{-1/2}(\gamma_0)} + \|h_3\|_{H^{1/2}(\gamma_0)}) & \textit{soft case}, \\ C_1 e^{-C_2/t}(\|h_s\|_{H^{1/2}(\gamma_0)} + \|h_3\|_{H^{1/2}(\gamma_0)}) & \textit{hard case}. \end{cases}$$

*Proof.* Let $P_0 = \left\{ \vec{x} \in P \mid \operatorname{dist}(\vec{x}, \Gamma) \geq L/2 \right\}$ be an interior domain of $P$. Then

$$\|\bar{\boldsymbol{\rho}} - \boldsymbol{\rho}\|_{L^2(Q)} = \|(\chi - 1)\boldsymbol{\rho}\|_{L^2(P_0 \cap Q)}.$$

The corollary follows from (3.8.11), (3.8.14), (3.8.17), and Lemma 3.8.3. □

**Lemma 3.8.5.** *Let* $\boldsymbol{N}^{(2)}$ *be as in* (3.8.9). *Then there exists a constant* $C$ *independent of* $t$ *such that*

$$\|\boldsymbol{N}^{(2)}\|_t' \leq \begin{cases} C\left(t^{3/2}\|f\|_{H^{-1/2}(\gamma_0)} + t^{-1/2}\|h_3\|_{H^{1/2}(\gamma_0)}\right) & \textit{soft case}, \\ C\left(t^{1/2}\|h_s\|_{H^{1/2}(\gamma_0)} + t^{-1/2}\|h_3\|_{H^{1/2}(\gamma_0)}\right) & \textit{hard case}. \end{cases}$$

*Proof.* By (3.8.16),

$$\|\boldsymbol{N}^{(2)}\|_t' \leq C|||\xi\boldsymbol{\rho}|||_t.$$

By the expression for $\boldsymbol{\rho}$ in (3.3.4) and Theorems 3.6.6 and 3.6.9,

$$\|\xi\rho_{nn}\|_{L^2(Q)} + \|\xi\rho_{ns}\|_{L^2(Q)} + \|\xi\rho_{ss}\|_{L^2(Q)} + t\|\xi\rho_{n3}\|_{L^2(Q)} + t\|\xi\rho_{s3}\|_{L^2(Q)} \\ + t^2\|\xi\rho_{33}\|_{L^2(Q)} \begin{cases} C\left(t^{3/2}\|f\|_{H^{-1/2}(\gamma_0)} + t^{-1/2}\|h_3\|_{H^{1/2}(\gamma_0)}\right) & \text{soft case}, \\ C\left(t^{1/2}\|h_s\|_{H^{1/2}(\gamma_0)} + t^{-1/2}\|h_3\|_{H^{1/2}(\gamma_0)}\right) & \text{hard case}. \end{cases}$$

The left hand side of this inequality is equivalent to $|||\xi\boldsymbol{\rho}|||_t$, the lemma follows. □

Now we estimate $|||\boldsymbol{L}^{(2)}|||_t'$, $|||\boldsymbol{L}^{(3)}|||_t'$. First we notice that the following identity:

$$\boldsymbol{\varepsilon}(\vec{y}) = \frac{\partial\psi}{\partial\xi}\vec{n}\vec{n}^T + \frac{1}{2}\frac{\partial\phi}{\partial\xi}(\vec{n}\vec{s}^T + \vec{s}\vec{n}^T) + \frac{1}{2}\left(\frac{\partial\mu}{\partial\xi} + \frac{\partial\psi}{\partial x_3}\right)(\vec{n}\vec{e}_3^T + \vec{e}_3\vec{n}^T) \\ + \frac{1}{2}\frac{\partial\phi}{\partial x_3}(\vec{s}\vec{e}_3^T + \vec{e}_3\vec{s}^T) + \frac{\partial\mu}{\partial x_3}\vec{e}_3\vec{e}_3^T + \boldsymbol{L}^{(3)}.$$

Thus, by (3.8.12), we obtain

$$\boldsymbol{L}^{(2)} = (\chi - 1)\left[\frac{\partial\psi}{\partial\xi}\vec{n}\vec{n}^T + \frac{1}{2}\frac{\partial\phi}{\partial\xi}(\vec{n}\vec{s}^T + \vec{s}\vec{n}^T) + \frac{1}{2}\left(\frac{\partial\mu}{\partial\xi} + \frac{\partial\psi}{\partial x_3}\right)(\vec{n}\vec{e}_3^T + \vec{e}_3\vec{n}^T)\right.$$
$$\left. + \frac{\partial\mu}{\partial x_3}\vec{e}_3\vec{e}_3^T + \boldsymbol{L}^{(3)}\right] + \frac{d\chi}{d\xi}\left[\psi\vec{n}\vec{n}^T + \frac{1}{2}\phi(\vec{n}\vec{s}^T + \vec{s}\vec{n}^T) + \frac{1}{2}\mu(\vec{n}\vec{e}_3^T + \vec{e}_3\vec{n}^T)\right]. \tag{3.8.19}$$

From Theorems 3.6.2 and 3.6.8 there exist positive constants $C_1$ and $C_2$ independent of $t$ such that

$$\begin{aligned}
&\left\|(1-\chi)\frac{\partial\psi}{\partial\xi}\right\|_{L^2(\omega)} + \left\|(1-\chi)\frac{\partial\phi}{\partial\xi}\right\|_{L^2(\omega)} + \left\|(1-\chi)\left(\frac{\partial\mu}{\partial\xi} + \frac{\partial\psi}{\partial x_3}\right)\right\|_{L^2(\omega)} \\
&\quad + \left\|(1-\chi)\frac{\partial\phi}{\partial x_3}\right\|_{L^2(\omega)} + \left\|(1-\chi)\frac{\partial\mu}{\partial x_3}\right\|_{L^2(\omega)} \\
&\quad \le \begin{cases} C_1 e^{-C_2/t}(\|f\|_{H^{-1/2}(\gamma_0)} + \|h_3\|_{H^{1/2}(\gamma_0)}) & \text{soft case,} \\ C_1 e^{-C_2/t}(\|h_s\|_{H^{1/2}(\gamma_0)} + \|h_3\|_{H^{1/2}(\gamma_0)}) & \text{hard case.} \end{cases}
\end{aligned} \tag{3.8.20}$$

By (3.8.19),

$$\begin{aligned}
&|||(\chi-1)L^{(3)}|||_t' + \frac{d\chi}{d\xi}|||\left[\psi\vec{n}\vec{n}^T + \frac{1}{2}\phi(\vec{n}\vec{s}^T + \vec{s}\vec{n}^T) + \frac{1}{2}\mu(\vec{n}\vec{e}_3^T + \vec{e}_3\vec{n}^T)\right]|||_t' \\
&\le |||L^{(3)}|||_t' + \frac{d\chi}{d\xi}\left(\|\psi\|_{L^2(Q)} + \|\phi\|_{L^2(Q)} + \frac{1}{t}\|\mu\|_{L^2(Q)}\right) \\
&\le C\left(\|\psi\|_{L^2(Q)} + \|\phi\|_{L^2(Q)} + \frac{1}{t}\|\mu\|_{L^2(Q)} + \left\|\frac{\partial\psi}{\partial\theta}\right\|_{L^2(Q)}\right. \\
&\quad \left. + \left\|\frac{\partial\phi}{\partial\theta}\right\|_{L^2(Q)} + \frac{1}{t}\left\|\frac{\partial\mu}{\partial\theta}\right\|_{L^2(Q)}\right).
\end{aligned} \tag{3.8.21}$$

Together this shows that $|||\boldsymbol{L}^{(2)}|||_t'$ is bounded by the right hand sides of (3.8.20) and (3.8.21).

Notice that bound for $\|L^{(3)}\|_t'$ is also given in (3.8.21). Therefore we have the following estimates.

**Lemma 3.8.6.** *Let $\boldsymbol{L}^{(2)}$, $\boldsymbol{L}^{(3)}$ be as in* (3.8.5), (3.8.6) *respectively. Then there exists a constant $C$ independent of $t$ such that*

$$|||\boldsymbol{L}^{(2)}|||_t' + |||\boldsymbol{L}^{(3)}|||_t' \le \begin{cases} C\left(t^{3/2}\|f\|_{H^{1/2}(\Gamma_0)} + t^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{soft case,} \\ C\left(t^{1/2}\|h_s\|_{H^{3/2}(\Gamma_0)} + t^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{hard case.} \end{cases}$$

*Proof.* By Theorems 3.6.1 and 3.7.1, we obtain

$$\|\psi\|_{L^2(Q)} + \|\phi\|_{L^2(Q)} + \frac{1}{t}\|\mu\|_{L^2(Q)} \leq \begin{cases} C\left(t^{3/2}\|f\|_{H^{-1/2}(\gamma_0)} + t^{-1}\|h_3\|_{H^{1/2}(\gamma_0)}\right) & \text{soft case,} \\ C\left(t^{1/2}\|h_s\|_{H^{1/2}(\gamma_0)} + t^{-1}\|h_3\|_{H^{1/2}(\gamma_0)}\right) & \text{hard case.} \end{cases} \tag{3.8.22}$$

Since $\theta$ enters as a parameter, and the bound in (3.8.22) is independent of $\theta$, then

$$\left\|\frac{\partial\psi}{\partial\theta}\right\|_{L^2(Q)} + \left\|\frac{\partial\phi}{\partial\theta}\right\|_{L^2(Q)} + \frac{1}{t}\left\|\frac{\partial\mu}{\partial\theta}\right\|_{L^2(Q)} \leq \begin{cases} C\left(t^{3/2}\left\|\frac{\partial f}{\partial\theta}\right\|_{H^{-1/2}(\Gamma_0)} + t^{-1}\left\|\frac{\partial h_3}{\partial\theta}\right\|_{H^{1/2}(\Gamma_0)}\right) & \text{soft case,} \\ C\left(t^{1/2}\left\|\frac{\partial h_s}{\partial\theta}\right\|_{H^{1/2}(\Gamma_0)} + t^{-1}\left\|\frac{\partial h_3}{\partial\theta}\right\|_{H^{1/2}(\Gamma_0)}\right) & \text{hard case.} \end{cases} \tag{3.8.23}$$

The lemma follows from (3.8.22) and (3.8.23)

Finally, we estimate $\|\boldsymbol{N}^{(1)}\|_t'$.

**Lemma 3.8.7.** *Let $\boldsymbol{N}^{(1)}$ be as in (3.8.8). Then there exists a constant $C$ independent of $t$ such that*

$$\|\boldsymbol{N}^{(1)}\|_t' \leq \begin{cases} C\left(t^{1/2}\|f\|_{H^{1/2}(\Gamma_0)} + t^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{soft case,} \\ C\left(t^{-1/2}\|h_s\|_{H^{3/2}(\Gamma_0)} + t^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{hard case.} \end{cases}$$

*Proof.* From (3.8.15),

$$\langle \boldsymbol{N}^{(1)}, \vec{v}\rangle = \int_{-1/2}^{1/2}\int_0^S\int_0^L \left[\rho_{ns}\left(\frac{\partial v_n}{\partial\theta} + \frac{1}{R}v_s\right) + \rho_{ss}\left(\frac{\partial v_s}{\partial\theta} - \frac{1}{R}\right) + \rho_{3s}\frac{\partial v_s}{\partial\theta}\right]. \tag{3.8.24}$$

First, we consider the first term of the right hand side of (3.8.24). By (3.3.4) and Theorem 3.6.1,

$$\int_{-1/2}^{1/2}\int_0^S\int_0^L \rho_{ns}\left(\frac{\partial v_n}{\partial\theta} + \frac{1}{R}v_s\right) \leq C\|\rho_{ns}\|_{L^2(Q)}\|\vec{v}\|_t \leq \begin{cases} Ct^{1/2}\|f\|_{H^{-1/2}(\gamma_0)}\|\vec{v}\|_t & \text{soft case,} \\ Ct^{-1/2}\|h_s\|_{H^{1/2}(\gamma_0)}\|\vec{v}\|_t & \text{hard case.} \end{cases} \tag{3.8.25}$$

Next, by (3.2.4) $v_3 = 0$ on $\Gamma_0$, and so $\chi v_3 = 0$ on $\Gamma_0 \cup \Gamma_L$. Thus,

$$\begin{aligned}
&\int_{-1/2}^{1/2}\int_0^S\int_0^L \rho_{s3}\frac{\partial v_3}{\partial\theta} = \int_{-1/2}^{1/2}\int_0^S\int_0^L \frac{\partial\rho_{s3}}{\partial\theta}[\chi v_3 + (1-\chi)v_3] \\
&\quad\le \int_0^S \left\|\frac{\partial\rho_{s3}}{\partial\theta}\right\|_{H^{-1,0}(\omega)} \|\chi v_3\|_{H^{1,0}(\omega)}\,d\theta + \left\|(1-\chi)\frac{\partial\rho_{s3}}{\partial\theta}\right\|_{L^2(Q)} \|v_3\|_{L^2(Q)} \\
&\quad\le C\left(\left\|\frac{\partial\rho_{s3}}{\partial\theta}\right\|_{H^{-1,0}(\omega)} + \left\|(1-\chi)\frac{\partial\rho_{s3}}{\partial\theta}\right\|_{L^2(Q)}\right)\|v_3\|_{H^1(Q)} \\
&\quad\le \begin{cases} Ct^{1/2}\|f\|_{H^{1/2}(\Gamma_0)}\|\vec{v}\|_t & \text{soft case,} \\ Ct^{-1/2}\|h_s\|_{H^{3/2}(\Gamma_0)}\|\vec{v}\|_t & \text{hard case.}\end{cases}
\end{aligned} \tag{3.8.26}$$

The last inequality follows from (3.3.4) and Theorems 3.6.2, 3.6.6.

Finally, we consider the second term of the right hand side of (3.8.24):

$$\begin{aligned}
&\int_{-1/2}^{1/2}\int_0^S\int_0^L \rho_{ss}\left(\frac{\partial v_s}{\partial\theta} - \frac{1}{R-\xi}v_n\right) \\
&\quad= \int_{-1/2}^{1/2}\int_0^S\int_0^L \frac{\partial\rho_{ss}}{\partial\theta}v_s - \int_{-1/2}^{1/2}\int_0^S\int_0^L \frac{1}{R}\rho_{ss}v_n.
\end{aligned} \tag{3.8.27}$$

Define

$$\tilde{\rho}(\theta, x_3) = \frac{1}{L}\int_0^L \rho_{ss}(\xi,\theta,x_3)\,d\xi.$$

We have the following inequalities.

$$\begin{aligned}
&\int_{-1/2}^{1/2}\int_0^S\int_0^L \frac{1}{R}\rho_{ss}v_n = \int_{-1/2}^{1/2}\int_0^S\int_0^L \frac{1}{R}(\rho_{ss}-\tilde{\rho})v_n + \int_{-1/2}^{1/2}\int_0^S\int_0^L \frac{1}{R}\tilde{\rho}v_n \\
&\quad\le \int_0^S \frac{1}{R}\int_{-1/2}^{1/2}\int_0^L (\rho_{ss}-\tilde{\rho})v_n + C\|\tilde{\rho}\|_{L^2(Q)}\|v_n\|_{L^2(Q)} \\
&\quad= \int_0^S \frac{1}{R}\int_{-1/2}^{1/2}\int_0^L \left[\int_\xi^L (\rho_{ss}-\tilde{\rho})\right]\frac{\partial v_n}{\partial\xi} + C\|\tilde{\rho}\|_{L^2(Q)}\|v_n\|_{L^2(Q)} \\
&\quad\le C\int_0^S \frac{1}{R}\left[\left\|\int_\xi^L (\rho_{ss}-\tilde{\rho})\right\|_{L^2(\omega)} \|v_n\|_{H^{1,0}(\omega)}\right] d\theta + C\|\tilde{\rho}\|_{L^2(Q)}\|v_n\|_{L^2(Q)} \\
&\quad\le C\left(\left\|\int_\xi^L (\rho_{ss}-\tilde{\rho})\right\|_{L^2(\omega)} \|v_n\|_{H^{1,0}(\omega)} + \|\tilde{\rho}\|_{L^2(Q)}\|v_n\|_{L^2(Q)}\right)
\end{aligned}$$

$$\leq C\left(\left\|\int_{\xi}^{L}\rho_{ss}\right\|_{L^2(\omega)}+\left\|\int_{\xi}^{L}\tilde{\rho}\right\|_{L^2(\omega)}+\|\tilde{\rho}\|_{L^2(\omega)}\right)\|\vec{v}\|_t. \tag{3.8.28}$$

By (3.3.4),

$$\rho_{ss}=\frac{E\nu}{(1+\nu)(1-2\nu)t^2}\left(\frac{\partial\mu}{\partial x_3}+t^2\frac{\partial\psi}{\partial\xi}\right). \tag{3.8.29}$$

By (3.8.29), Theorems 3.6.7 and 3.6.8, there exist positive constants $C$ and $\lambda$ such that

$$\|\rho_{ss}\|_{L^2(\tilde{\omega}_\alpha)}\leq Ct^{-3/2}e^{-\lambda\alpha/2t}\|h_3\|_{H^{1/2}(\gamma_0)},\qquad\text{for all } t\leq\alpha\leq L,\ \text{and } \alpha=0. \tag{3.8.30}$$

By (3.8.30) and Lemma 3.6.5,

$$\begin{aligned}&\left\|\int_{\xi}^{L}\rho_{ss}\right\|_{L^2(\omega)}\leq Ct^{-1/2}\|h_3\|_{H^{1/2}(\gamma_0)},\\&\|\tilde{\rho}\|_{L^2(\omega)}\leq Ct^{-1}\|h_3\|_{H^{1/2}(\gamma_0)},\\&\left\|\int_{\xi}^{L}\tilde{\rho}\right\|_{L^2(\omega)}\leq Ct^{-1}\|h_3\|_{H^{1/2}(\gamma_0)}.\end{aligned} \tag{3.8.31}$$

It follows from (3.8.28) and (3.8.31) that

$$\int_{-1/2}^{1/2}\int_0^S\int_0^L\frac{1}{R}\rho_{ss}v_n\leq Ct^{-1}\|h_3\|_{H^{1/2}(\gamma_0)}\|\vec{v}\|_t. \tag{3.8.32}$$

The same argument as above leads to

$$\int_{-1/2}^{1/2}\int_0^S\int_0^L\frac{\partial\rho_{ss}}{\partial\theta}v_s\leq Ct^{-1}\left\|\frac{\partial h_3}{\partial\theta}\right\|_{H^{1/2}(\Gamma_0)}\|\vec{v}\|_t\leq Ct^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\|\vec{v}\|_t. \tag{3.8.33}$$

From (3.8.27), (3.8.32) and (3.8.33), we obtain

$$\int_{-1/2}^{1/2}\int_0^S\int_0^L\rho_{ss}\left(\frac{\partial v_s}{\partial\theta}\frac{1}{1-\xi/R}-\frac{v_n}{R-\xi}\right)J\leq Ct^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\|\vec{v}\|_t. \tag{3.8.34}$$

The lemma then follows from (3.8.25), (3.8.26) and (3.8.34). □

**Theorem 3.8.8.** *Let* $(\boldsymbol{\eta}, \vec{z})$ *be as in* (3.8.1), $0 < t \leq 1$. *Then*

$$|||\boldsymbol{\eta}|||_t + \|\vec{z}\|_t \leq \begin{cases} C\left(t^{1/2}\|f\|_{H^{1/2}(\Gamma_0)} + t^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{soft case,} \\ C\left(t^{-1/2}\|h_s\|_{H^{3/2}(\Gamma_0)} + t^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{hard case.} \end{cases}$$

*Proof.* By Corollary 3.8.2, the theorem follows from Corollary 3.8.4, Lemma 3.8.5, 3.8.6, and 3.8.7. □

**Global and local estimates for** $\boldsymbol{\sigma}^t$**.**

**Theorem 3.8.9.** *Let* $\boldsymbol{\sigma}^t$ *be the solution to* (3.2.8). *Then there exists a constant* $C$ *independent of* $t$ *such that*

$$|||\boldsymbol{\sigma}^t|||_t \leq \begin{cases} C\left(t^{1/2}\|f\|_{H^{1/2}(\Gamma_0)} + t^{-3/2}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{soft case,} \\ C\left(t^{-1/2}\|h_s\|_{H^{3/2}(\Gamma_0)} + t^{-3/2}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{hard case.} \end{cases} \tag{3.8.35}$$

$$|||\boldsymbol{\sigma}^t|||_{t,P_0} \leq \begin{cases} C\left(t^{1/2}\|f\|_{H^{1/2}(\Gamma_0)} + t^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{soft case,} \\ C\left(t^{-1/2}\|h_s\|_{H^{3/2}(\Gamma_0)} + t^{-1}\|h_3\|_{H^{3/2}(\Gamma_0)}\right) & \textit{hard case} \end{cases} \tag{3.8.36}$$

*Moreover, for the hard simply supported plate there exists another positive constant* $C'$ *independent of* $t$ *such that*

$$|||\boldsymbol{\sigma}^t|||_t \geq C'\left(t^{-1/2}\|h_s\|_{H^{1/2}(\gamma_0)} + t^{-3/2}\|h_3\|_{H^{1/2}(\gamma_0)}\right). \tag{3.8.37}$$

*Proof.* By the definition of $\boldsymbol{\eta}$,

$$\boldsymbol{\sigma}^t = \boldsymbol{\eta} + \bar{\boldsymbol{\rho}}. \tag{3.8.38}$$

By (3.3.4), Theorem 3.6.1 and 3.6.7, there exist positive constants $C_1$ and $C_2$ such that

$$\begin{aligned} C_1\left(t^{1/2}\|f\|_{H^{-1/2}(\gamma_0)} + t^{-3/2}\|h_3\|_{H^{1/2}(\gamma_0)}\right) &\leq |||\chi_Q\boldsymbol{\rho}|||_t \\ &\leq C_2\left(t^{1/2}\|f\|_{H^{-1/2}(\gamma_0)} + t^{-3/2}\|h_3\|_{H^{1/2}(\gamma_0)}\right) \quad \text{soft case,} \\ C_1\left(t^{-1/2}\|h_s\|_{H^{1/2}(\gamma_0)} + t^{-3/2}\|h_3\|_{H^{1/2}(\gamma_0)}\right) &\leq |||\chi_Q\boldsymbol{\rho}|||_t \\ &\leq C_2\left(t^{-1/2}\|h_s\|_{H^{1/2}(\gamma_0)} + t^{-3/2}\|h_3\|_{H^{1/2}(\gamma_0)}\right) \quad \text{hard case.} \end{aligned} \tag{3.8.39}$$

It is easy to see that there exist also constants $C_1'$ and $C_2'$ independent of $t$ such that

$$C_1' |||\chi_Q \boldsymbol{\rho}|||_t \leq |||\bar{\boldsymbol{\rho}}|||_t \leq C_2' |||\chi_Q \boldsymbol{\rho}|||_t. \tag{3.8.40}$$

Using the triangle inequality, (3.8.35) and (3.8.37) follows from (3.8.38), (3.8.39), (3.8.40) and Theorem 3.8.8.

For the interior estimates, by Lemma 3.8.3, there exists positive constants $C_1$ and $C_2$ such that

$$|||\bar{\boldsymbol{\rho}}|||_{t,P_0} \leq \begin{cases} C_1 e^{-C_2/t} \left( \|f\|_{H^{-1/2}(\gamma_0)} + \|h_3\|_{H^{3/2}(\gamma_0)} \right) & \text{soft case,} \\ C_1 e^{-C_2/t} \left( \|h_s\|_{H^{1/2}(\gamma_0)} + \|h_3\|_{H^{1/2}(\gamma_0)} \right) & \text{hard case.} \end{cases}$$

Thus the order of $|||\bar{\boldsymbol{\rho}}_{\alpha\beta}|||_{t,P_0}$ is of the same order as that of $|||\boldsymbol{\eta}|||_{t,P_0}$. The inequality (3.8.36) then follows from Theorem 3.8.8. $\square$

## <u>Sec 3.9. Convergence Results</u>

We return to the domain $P^t$ and discuss the order of the three-dimensional boundary corrector $(\boldsymbol{\sigma}^{\mathrm{c}}, \vec{u}^{\mathrm{c}})$ defined in (3.1.1). Note that $\vec{u}^{\mathrm{c}}$ corrects the boundary value of $\vec{u}^{\mathrm{k}}$, which can be either $\vec{u}^{\mathrm{m}}$ or $\vec{u}^{\mathrm{s}}$ in (2.4.2) or (2.4.3) respectively. As we shall see, Theorem 3.8.10 implies that the orders of $\|\boldsymbol{\sigma}_c\|_{L^2(P^t)}$ and $\|\boldsymbol{\sigma}_c\|_{L^2(P_0^t)}$ do not exceed $O(t^{5/2})$. Thus $\vec{u}^{\mathrm{k}} = \vec{u}^{\mathrm{m}}$ is a more appropriate choice. If we choose $\vec{u}^{\mathrm{k}} = \vec{u}^{\mathrm{s}}$, we will get the same order estimates but to bound the term $\|h_s\|_{H^{3/2}(\Gamma_0)}$, by (2.4.3), we need more regularity assumption on the data $g$:

$$\|h_s\|_{H^{3/2}(\Gamma_0)} \leq Ct^3 \|w\|_{H^5(\Omega)} \leq Ct^3 \|g\|_{H^1(\Omega)}.$$

where $w$ is the Kirchhoff plate solution, $g$ the scaled loading of the three-dimensional plate. Both $w$ and $g$ are independent of $t$.

Now, with $\vec{u}^{\mathrm{k}} = \vec{u}^{\mathrm{m}}$, by (2.4.2),

$$h_s = 0. \tag{3.9.1}$$

By (2.4.2) and the trace theorem, there exists a constant $C$ independent of $t$ such that

$$\|h_3\|_{H^{3/2}(\Gamma_0)} + t^2\|f\|_{H^{1/2}(\Gamma_0)} \le Ct^3\|w\|_{H^4(\Omega)} \le Ct^3\|g\|_{L^2(\Omega)}.$$

We now give bounds using data $g$ instead of $h_3$, $h_s$, and $f$.

**Theorem 3.9.1.** *Let* $(\boldsymbol{\sigma}^{\mathrm{c}}, \vec{u}^{\mathrm{c}})$ *be the solution to* (3.1.1) *and* (3.1.2), $0 < t \le 1$. *Then there exists a constant* $C$ *independent of* $t$ *such that*

$$\|\boldsymbol{\sigma}^{\mathrm{c}}\|_E \le Ct^2\|g\|_{L^2(\Omega)},$$

*and*

$$\|\boldsymbol{\sigma}^{\mathrm{c}}\|_{E,P_0^t} \le \begin{cases} Ct^2\|g\|_{L^2(\Omega)} & \textit{soft case}, \\ Ct^{5/2}\|g\|_{L^2(\Omega)} & \textit{hard case}, \end{cases}$$

*Moreover, for the hard simply supported plate, there exits a constant* $C'$ *independent of* $t$ *such that*

$$\|\boldsymbol{\sigma}^{\mathrm{c}}\|_E \ge C't^2\|g\|_{L^2(\Omega)}.$$

*Proof.* The theorem follows from the definitions of $\|\cdot\|_E$ and $\|\cdot\|_{E,P_0^t}$ in (1.11), Theorem 3.8.9, (3.9.1), and the scalings in (3.2.1) and (3.2.2). □

Now with the order of the boundary correctors known, we derive the convergence results introduced in the first chapter.

**Theorem 3.9.2.** *Let* $\boldsymbol{\sigma}$ *and* $\vec{u}$ *be defined by either the soft simply supported plate problem* (1.1)–(1.4), (1.5), (1.8) *or the hard simply supported plate problem* (1.1)–(1.4), (1.6), *and let* $\boldsymbol{\sigma}^{\mathrm{k}}$ *and* $\vec{u}^{\mathrm{k}}$ *be the Kirchhoff approximations defined by* (1.9), *(1.10), (2.2.2), and (2.2.6). Then there exists a constant $C$ depending only on the domain $\Omega$ such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E + |||\vec{u} - \vec{u}^{\mathrm{k}}||| \le Ct^2 \|g\|_{L^2(\Omega)}.$$

*Moreover, for the hard simply supported plate there exits another constant $C'$ depending only on the domain $\Omega$ such that*

$$|||\vec{u} - \vec{u}^{\mathrm{k}}||| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{k}}\|_E \ge C' t^2 \|g\|_{L^2(\Omega)}.$$

*Proof.* The theorem follows from Theorem 2.4.1, the scaling in (3.2.1), and Theorem 3.9.1. □

By (2.2.2) and (2.2.6), it is easy to see that there exists constants $C_1$ and $C_2$ such that

$$C_1 t^{3/2} \|g\|_{L^2(\Omega)} \le |||\vec{u}^{\mathrm{k}}||| + \|\boldsymbol{\sigma}^{\mathrm{k}}\|_E \le C_2 t^{3/2} \|g\|_{L^2(\Omega)}. \tag{3.9.2}$$

Thus, we have the following result.

**Theorem 3.9.3.** *Let $\boldsymbol{\sigma}$, $\vec{u}$ be the same as in Theorem 3.9.2. Then there exists positive constants $C_1$ and $C_2$ independent of $t$ such that*

$$C_1 t^{3/2} \|g\|_{L^2(\Omega)} \le |||\vec{u}||| + \|\boldsymbol{\sigma}\|_E \le C_2 t^{3/2} \|g\|_{L^2(\Omega)}.$$

*Proof.* The result follows immediately from Theorem 3.9.2 and (3.9.2). □

From Theorem 3.9.2 and 3.9.3, the following convergence result immediately follows.

**Theorem 3.9.4.** *The global convergence rate for both the soft and hard simply supported plate is $O(t^{1/2})$. More precisely, let $\boldsymbol{\sigma}$, $\vec{u}$, $\boldsymbol{\sigma}^{\mathrm{k}}$, and $\vec{u}^{\mathrm{k}}$ be as in Theorem 3.9.2. Then there is a constant $C$ independent of $t$ such that*

$$\frac{|||\vec{u}-\vec{u}^{\mathrm{k}}|||}{|||\vec{u}|||}+\frac{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{k}}\|_E}{\|\boldsymbol{\sigma}\|_E}\le Ct^{1/2}.$$

*Moreover, for the hard simply supported plate the convergence rate of $O(t^{1/2})$ is sharp, i.e., there exists a positive constant $C'$ independent of $t$ such that*

$$\frac{|||\vec{u}-\vec{u}^{\mathrm{k}}|||}{|||\vec{u}|||}+\frac{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{k}}\|_E}{\|\boldsymbol{\sigma}\|_E}\ge C't^{1/2}.$$

□

It is very likely that the $O(t^{1/2})$ convergence rate is sharp for the soft simply supported plate as well. For that plate, since the orders of the boundary data $h_3$ and $f$ are $O(t^3)$ and $O(t)$ respectively, then by Theorem 3.8.8, the order of $\|\boldsymbol{\eta}\|_{L^2(P)}$ is $O(t^{3/2})$, which is the same as that of $\|\bar{\boldsymbol{\rho}}\|_{L^2(P)}$. Thus our analysis fails to determine the order of $\|\boldsymbol{\sigma}^t\|_{L^2(P)}=\|\bar{\boldsymbol{\rho}}+\boldsymbol{\eta}\|_{L^2(P)}$. However, by (3.8.1), it is highly unlikely that all the lowest order terms of $\bar{\boldsymbol{\rho}}$ can cancel out with all the lowest order terms of $\boldsymbol{\eta}$. Thus our conjecture is that $\|\boldsymbol{\sigma}^t\|_{L^2(P)}$ is still of order $O(t^{3/2})$ for the soft simply supported plate. If that conjecture is true, then the $O(t^{1/2})$ global convergence rate is sharp for the soft simply supported plate.

**Theorem 3.9.5.** *The interior convergence rate for the soft simply supported plate is $O(t^{1/2})$, and for the hard simply supported plate is $O(t)$. More precisely, Let $\boldsymbol{\sigma}$, $\vec{u}$, $\boldsymbol{\sigma}^{\mathrm{k}}$, $\vec{u}^{\mathrm{k}}$ be as in Theorem 3.9.2. Then there exists a constant $C$ independent of $t$ such that*

$$|||\vec{u}-\vec{u}^{\mathrm{k}}|||_{P_0^t}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{k}}\|_{E,P_0^t}\le Ct^2\|g\|_{L^2(\Omega)}\qquad \textit{soft case},$$
$$|||\vec{u}-\vec{u}^{\mathrm{k}}|||_{P_0^t}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm{k}}\|_{E,P_0^t}\le Ct^{5/2}\|g\|_{L^2(\Omega)}\quad \textit{hard case}.$$

□

# Chapter Four

## SUMMARY

In this thesis we analyzed the accuracy of the Kirchhoff plate model as an approximation to the full system of three-dimensional linear elasticity, considering the cases of soft and of hard simply supported boundary conditions. The key results were energy norm estimates for this approximation, both global and restricted to an interior subdomain disjoint from the lateral boundary. In some cases we were able to prove the sharpness of our estimates.

The Kirchhoff plate solution is a scalar function defined on the midsurface of the plate. From this function we constructed approximations to the displacement and to the stress. These approximations are 3-vector-valued and $3 \times 3$-tensor-valued functions defined on the three-dimensional plate domain. Specifically we introduced modifications of expressions which had been developed earlier by Morgenstern and by Simmonds. Simmonds's expression is more accurate in some situations, but requires more regularity than Morgenstern's. Our modifications affect neither the accuracy nor the regularity, but were introduced to simplify the error analysis.

Note that the Kirchhoff model does not distinguish between the hard and soft simply supported plates: the same approximation is obtained for both. In fact the simply supported Kirchhoff plate is closer to the hard simply supported plate than to the more physically relevant soft simply supported plate. This is reflected in the final results, which give higher order interior convergence in the hard simply supported case.

The basis of the error analysis is the Prager–Synge theorem. Since neither of the approximations satisfy the lateral boundary conditions required by that theorem, a boundary corrector was introduced. Unlike in previous work, we defined the boundary corrector as the solution of a three-dimensional elasticity problem. The analysis then reduced to determining global and interior energy norm bounds on the boundary corrector. Our approach to this problem was strongly influenced by the work of Destuynder, but includes new elements as well, especially the explicit use of boundary-fitted coordinates and of Saint Venant's principle.

Since the asymptotic analysis of the three-dimensional problem defining the boundary corrector is too difficult, we first consider a simplified auxiliary problem which (it turns out) has similar asymptotic behavior. This auxiliary problem is obtained from the three-dimensional problem by restricting to a neighborhood of the lateral boundary of the plate, using boundary-fitted coordinates, neglecting the derivatives with respect to the tangential direction, and suppressing the Jacobian which arises from the change of coordinates. The auxiliary problem then decoupled into a two-dimensional Laplace-like problem and a two-dimensional elasticity-like problem, both parameterized by the tangential coordinate variable. Exponential decay properties of the solutions of these two problems were then obtained in the spirit of Saint Venant's principle, and from these properties we determined the global and interior orders of the solution to the auxiliary problem. Next we bounded the global energy norm of the difference between the three-dimensional boundary corrector and the solution to the auxiliary problem. Finally we combined these results to obtain the global and interior energy norm bounds for the boundary corrector.

With this approach we have proved that the known global convergence rate of $O(t^{1/2})$ for the hard simply supported plate with smooth boundary is sharp. (This is the rate of convergence of the relative energy norm error.) We have also derived the interior convergence rate of $O(t)$ for hard simply supported plate. To the best of our knowledge, both of these results are new. The same analysis applied to the soft simply supported plate gives both global and interior convergence rates of $O(t^{1/2})$. Our analysis strongly suggests, although does not definitively prove, that both these convergence rates are sharp as well.

The low orders of convergence in these results contrast with the second order convergence which we established for a periodic plate. This difference suggests the effect of boundary layers in determining the accuracy of the Kirchhoff model.

An interesting area for future investigation is the accuracy of the Reissner–Mindlin plate model. While estimates for the error in the Reissner–Mindlin model as an approximation to three-dimensional elasticity can be obtained using the results here and known results for the difference between the Reissner–Mindlin approximation and the Kirchhoff approximation, it may be possible to obtain sharper results in some cases by applying the approach here directly to the Reissner–Mindlin model. In particular it would be interesting to know whether the Reissner–Mindlin model can be used to obtain interior convergence of higher than $O(t^{1/2})$ for the soft simply supported plate.

# INDEX OF NOTATIONS

Vectors in $\mathbb{R}^3$ are denoted by Latin letters with arrows: $\vec{u}$, $\vec{v}$. Vectors in $\mathbb{R}^2$ are denoted by Latin letters with under-tildes: $\underset{\sim}{u}$, $\underset{\sim}{v}$. Tensors in $\mathbb{R}^{3\times 3}$ are denoted by bold Greek letters: $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$, $\boldsymbol{\tau}$, and tensors in $\mathbb{R}^{2\times 2}$ by Greek letters with double under-tildes: $\underset{\approx}{\tau}$, $\underset{\approx}{\delta}$. When used as indices, $i$ and $j$ range from 1 to 3 and $\alpha$, $\beta$, $\gamma$ range from 1 to 2.

The following notations are presented in the approximate order of their appearance.

| | |
|---|---|
| $P^t$ | three-dimensional plate domain |
| $t$ | plate thickness |
| $\Omega$ | midsurface of $P^t$, $P^t = \Omega \times (-1/2, 1/2)$ |
| $\Omega^t_+$ | top surface of $P^t$, $\Omega^t_+ = \Omega \times \{1/2\}$ |
| $\Omega^t_-$ | bottom surface of $P^t$, $\Omega^t_- = \Omega \times \{-1/2\}$ |
| $\Gamma^t$ | lateral boundary of $P^t$ |
| $\boldsymbol{\sigma}$ | three-dimensional stress tensor: $\boldsymbol{\sigma} : P^t \mapsto \mathbb{R}^{3\times 3}$ |
| $\vec{u}$ | three-dimensional displacement vector: $\vec{u} : P^t \mapsto \mathbb{R}^3$ |
| $E$ | Young's modulus |
| $\nu$ | Poisson's ratio |
| $\boldsymbol{\delta}$ | $3 \times 3$ identity matrix |
| $\underset{\approx}{\delta}$ | $2 \times 2$ identity matrix |
| $q_+$ | vertical surface traction density at the top surface |
| $q_-$ | vertical surface traction density at the bottom surface |
| $g$ | scaled vertical surface traction density at the top and bottom surfaces |

| | |
|---|---|
| $\vec{e}_3$ | unit vector, directed upward |
| $\vec{n}$ | unit normal vector to the lateral boundary, directed outward |
| $\vec{s}$ | unit in-plane tangential vector to the lateral boundary, directed counter-clockwise |
| $\mathcal{R}$ | space of in-plane rigid motions |
| $w$ | solution to the Kirchhoff plate equation |
| $A$ | compliance tensor, relating three-dimensional strain to three-dimensional stress |
| $\boldsymbol{\sigma}^k$ | Kirchhoff plate approximation to three-dimensional stress |
| $\vec{u}^{\mathrm{k}}$ | Kirchhoff plate approximation to $\vec{u}$ |
| $P_0^t$ | interior subdomain of $P^t$: $\Omega_0 \times (-t/2, t/2)$ |
| $\boldsymbol{\Sigma}$ | $\{\, \boldsymbol{\tau} \mid \tau_{ij} \in L^2(P^t),\ \tau_{ij} = \tau_{ji} \,\}$ |
| $\boldsymbol{V}$ | subspace of $\left\{ \vec{v} \mid v_i \in H^1(P^t) \right\}$; cf. (2.1.1) |
| $\lVert \cdot \rVert_E$ | energy norm on $\boldsymbol{\Sigma}$ |
| $\lvert\lvert\lvert \cdot \rvert\rvert\rvert$ | energy norm on $\boldsymbol{V}$ |
| $\tilde{\boldsymbol{\sigma}}$ | approximation to $\boldsymbol{\sigma}$ (used in Prager–Synge theorem) |
| $\tilde{\vec{u}}$ | approximation to $\vec{u}$ (used in Prager–Synge theorem) |
| $\boldsymbol{\sigma}^{\mathrm{c}}$ | boundary corrector for $\boldsymbol{\sigma}^{\mathrm{k}}$: $\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^{\mathrm{k}}$ |
| $\vec{u}^{\mathrm{c}}$ | boundary corrector for $\vec{u}^{\mathrm{k}}$: $\tilde{\vec{u}} - \vec{u}^{\mathrm{k}}$ |
| $\Delta$ | Laplacian |
| $\vec{\mathrm{grad}}$ | (3-vector) gradient of a scalar function |
| $\mathrm{div}$ | (scalar) divergence of a vector function |
| $\vec{\mathrm{div}}$ | (3-vector) divergence of a $3 \times 3$ tensor |
| $\underset{\sim}{\mathrm{grad}}$ | (2-vector) gradient of a scalar function |
| $\underset{\approx}{\mathrm{grad}}$ | ($2 \times 2$ tensor) gradient of a 2-vector function |
| $\vec{u}^{\mathrm{m}}$ | modification of Morgenstern's approximation to $\vec{u}$ |
| $\vec{u}^{\mathrm{k}}$ | modification of Simmonds's approximation to $\vec{u}$ |

| | |
|---|---|
| $h_3$ | vertical displacement on lateral boundary to be corrected |
| $h_s$ | tangential displacement on lateral boundary to be corrected |
| $f$ | tangential traction on lateral boundary to be corrected |
| $P$ | scaling of $P^t$, $P = \Omega \times (-1/2, 1/2)$ |
| $\Omega_+$ | top surface of $P$, $\Omega_+ = \Omega \times \{1/2\}$ |
| $\Omega_-$ | bottom surface of $P$, $\Omega_- = \Omega \times \{-1/2\}$ |
| $\Gamma$ | lateral boundary of $P$ |
| $\boldsymbol{\sigma}^t$ | scaling of $\boldsymbol{\sigma}^c$, $\boldsymbol{\sigma}^t : P \mapsto \mathbb{R}^{3\times 3}$ |
| $\boldsymbol{\Sigma}^t$ | $\{\, \boldsymbol{\tau} \mid \tau_{ij} \in L^2(P),\, \tau_{ij} = \tau_{ji} \,\}$ |
| $\boldsymbol{V}^t$ | subspace of $\left\{ \vec{v} \mid v_i \in H^1(P) \right\}$; cf. (3.2.4) |
| $a_i$, $b$ | bilinear forms for the scaled problem; cf. (3.2.8)–(3.2.9) |
| $\xi$ | independent variable in the direction of $-\vec{n}$ |
| $\theta$ | independent variable in the direction of $\vec{s}$ |
| $v_n$ | $-\vec{v} \cdot \vec{n}$ |
| $v_s$ | $\vec{v} \cdot \vec{s}$ |
| $\tau_{nn}$ | $\vec{n}^T \boldsymbol{\tau} \vec{n}$ |
| $\tau_{ns}$ | $-\vec{s}^T \boldsymbol{\tau} \vec{n}$ |
| $\tau_{ss}$ | $\vec{s}^T \boldsymbol{\tau} \vec{s}$ |
| $\tau_{n3}$ | $-\vec{n}^T \boldsymbol{\tau} \vec{e}_3$ |
| $\tau_{s3}$ | $\vec{s}^T \boldsymbol{\tau} \vec{e}_3$ |
| $\tau_{33}$ | $\vec{e}_3^T \boldsymbol{\tau} \vec{e}_3$ |
| $S$ | arclength of $\partial\Omega$ |
| $L$ | a constant less than half the smallest radius of curvature on $\partial\Omega$ |
| $Q$ | subset of $P$ consisting of points within distance $L$ of $\Gamma$ |
| $\Psi^+$ | $Q \cap \Omega_+$ |
| $\Psi^-$ | $Q \cap \Omega_-$ |
| $\Gamma_L$ | inner lateral boundary of $Q$ |

$R$ radius of curvature on $\partial\Omega$; $R = R(\theta)$

$J$ Jacobi determinant of transformation to the boundary-fitted coordinate system

$\boldsymbol{\rho}, \vec{y}$ solution to auxiliary problem of $Q$

$\boldsymbol{\Sigma}_Q$ $\{\, \boldsymbol{\tau} \mid \tau_{ij} \in L^2(Q),\ \tau_{ij} = \tau_{ji} \,\}$

$\boldsymbol{V}_Q$ subspace of $\left\{ \vec{v} \mid v_i \in H^1(Q) \right\}$; cf. (3.3.7)

$\chi_Q$ characteristic function of $Q$

$A_i^Q$, $B^Q$ bilinear form for the auxiliary problem; cf. (3.3.8)–(3.3.10)

$\gamma^{\pm}$ horizontal segment $(0, L) \times \{\pm 1/2\}$

$\gamma_\alpha$ vertical segment $\{\alpha\} \times (-1/2, 1/2)$

$\omega_{r_1,r_2}$ rectangle $(r_1, r_2) \times (-1/2, 1/2)$

$\omega_r$ rectangle $(0, r) \times (-1/2, 1/2)$

$\psi$, $\phi$, $\mu$ notations for $y_n$, $y_s$, $y_s$ respectively

$\chi$ cutoff function supported in $Q$, equal to 1 near $\Gamma$

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