Finite element differential forms

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Outline

- Background on the continuous problem
- Requirements for discretization
- Subspaces of finite element differential forms
- Application to elasticity

References

Finite element exterior calculus, homological techniques, and applications, Acta Numerica 2006, p. 1–155

> *"Any young (or not so young) mathematician who spends the time to master this paper will have tools that will be useful for his or her entire career."* — Math Reviews

Mixed finite element methods for linear elasticity with weakly imposed symmetry, Math. Comp. 2007

NEW! *Geometric decompositions and local bases for spaces finite element differential forms,* to appear in CMAME

everything is at http://umn.edu/∼arnold

Background on the continuous problem

- Exterior calculus
- The de Rham complex
- Hodge theory and the Hodge Laplacian

Differential forms

Let Ω be a domain in $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ smooth.

$\forall x \in \Omega$, $df_x : \mathbb{R}^n \to \mathbb{R}$ is a linear map: $df_x(v) = v \cdot \nabla f(x)$.

(More generally, on any smooth manifold, $df_x : T_x\Omega \to \mathbb{R}$ is linear)

f is a 0-form, df is a 1-form, $\Lambda^0(\Omega)$, $\Lambda^1(\Omega)$

ω is a k-form $\iff \forall x$, ω_x is an alternating k-linear form on $\mathbb{R}^n$ ($T_x\Omega$)

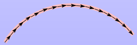
$dx^1, \ldots, dx^n$: standard dual basis of $\mathbb{R}^n \implies$
$dx^1 \wedge \cdots \wedge dx^k$, $i_1 < \cdots < i_k$, basis for alternating k-linear forms.

$$\omega \in \Lambda^k(\Omega) \iff \omega = \sum f_{i_1 \cdots i_k}\, dx^1 \wedge \cdots \wedge dx^k.$$

A k-form has $\binom{n}{k}$ coefficients (so $\Lambda^k(\Omega) = 0$ for $k > n$).

Exterior calculus

- Exterior derivative. A k-form ω can be differentiated to get a $(k+1)$-form $d^k\omega$: take the directional derivative of $\omega_x(v_1, \ldots, v_k)$ in the direction v_{k+1} and skew-symmetrize.
- $d^{k+1} \circ d^k = 0$, i.e., $\mathfrak{B}^k := \operatorname{range}(d^{k-1}) \subset \mathfrak{Z}^k := \ker(d^k)$
- If $F : \Omega \to \Omega'$, the pullback F^* maps $\Lambda^k(\Omega') \to \Lambda^k(\Omega)$:
  $$(F^*\omega)_x(v_1, \ldots, v_k) = \omega_{F(x)}(dF_x v_1, \ldots, dF_x v_k)$$
- If $S \subset \Omega$ is a submanifold, the pullback of the inclusion is the trace, $\operatorname{tr} : \Lambda^k(\Omega) \to \Lambda^k(S)$. This is *not* just the restriction.
- There is a natural integral of a k-form ω over a k-dimensional surface: $\int_S \omega \in \mathbb{R}$



- Stokes theorem: $\displaystyle\int_\Omega d\omega = \int_{\partial\Omega} \operatorname{tr}\omega, \quad \omega \in \Lambda^{n-1}(\Omega)$

The de Rham complex

We work in a Hilbert space setting.

$d = d^k$ is a *closed densely-defined* unbounded operator $L^2\Lambda^k(\Omega) \to L^2\Lambda^{k+1}(\Omega)$ with *closed range*.

Its domain is $H\Lambda^k(\Omega) = \{\,\omega \in L^2\Lambda^k(\Omega) \,|\, d\omega \in L^2\Lambda^{k+1}(\Omega)\,\}$, a Hilbert space with the graph norm $\|\omega\|^2_{H\Lambda} = \|\omega\|^2 + \|d\omega\|^2$.

They connect to form the de Rham complex:

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d^0} H\Lambda^1(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H\Lambda^n(\Omega) \to 0$$

The kth de Rham cohomology group is $\mathfrak{Z}^k/\mathfrak{B}^k$.

Since we are in a Hilbert setting, the cohomology space is isomorphic to the space of harmonic k-forms:

$$\mathfrak{H}^k := \{\,\omega \in \mathfrak{Z}^k \,|\, \omega \perp \mathfrak{B}^k\,\}$$

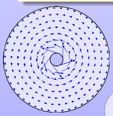
The case of $\Omega \subset \mathbb{R}^3$

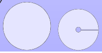
For $\Omega \subset \mathbb{R}^3$, the de Rham complex boils down to

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

scalar fn — vector field — vector field — scalar fn

> "Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area."
>
> – James Clerk Maxwell, *Treatise on Electricity & Magnetism*, 1891





$$\dim(\mathfrak{H}^k) = \begin{cases} \#\text{ components of } \Omega, & k = 0 \\ \#\text{ tunnels thru } \Omega, & k = 1 \\ \#\text{ voids in } \Omega, & k = 2 \\ 0, & k = 3 \end{cases}$$

For Ω contractible (e.g., convex)
$\mathfrak{H}^0 = \mathbb{R}$, $\mathfrak{H}^k = 0$, $k > 0$

Hodge theory

Using the inner product, we have:

- Hodge decomposition: $L^2\Lambda^k(\Omega) = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}$
- Poincaré's inequality: $\|\omega\|_{L^2} \le c\,\|d\omega\|_{L^2}, \quad \omega \in H\Lambda^k, \omega \perp \mathfrak{Z}^k$
- adjoint: $d^* : L^2\Lambda^{k+1}(\Omega) \to L^2\Lambda^k(\Omega)$, closed densely-defined
- $\mathfrak{H} = \ker(d) \cap \ker(d^*)$
- Hodge Laplace problem:
  Given $f \in \Lambda^k$ $(0 \le k \le n)$, find $u \in \Lambda^k$ such that
  $$(d^*d + d\,d^*)u = f \qquad \text{plus BC}$$
- The harmonic functions determine well-posedness:
  (1) $\exists u \iff f \perp \mathfrak{H}^k$, (2) u is determined only mod $\mathfrak{H}^k$

Naive weak formulation of the Hodge Laplacian

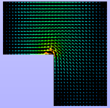
An obvious weak formulation (in the case of no harmonic forms):
Find $u \in H\Lambda^k \cap H^*\Lambda^k$ such that

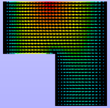
$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle, \quad v \in H\Lambda^k \cap H^*\Lambda^k$$

Equivalently, $\frac{1}{2}\|du\|^2 + \frac{1}{2}\|d^*u\|^2 - \langle f, u\rangle \to \textit{minimum}$

This formulation is problematic for discretization.
Lagrange elements, or any conforming finite elements, converge...
but not to the solution!

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f \text{ (with natural BC)}$$





Mixed formulation of the Hodge Laplacian

Given $f \in L^2\Lambda^k(\Omega)$, find $\sigma \in H\Lambda^{k-1}$, $u \in H\Lambda^k$, $p \in \mathfrak{H}^k$:

$$\begin{aligned}
&\langle \sigma, \tau\rangle - \langle d\tau, u\rangle = 0 && \forall \tau \in H\Lambda^{k-1}\\
&\langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle = \langle f, v\rangle && \forall v \in H\Lambda^k\\
&\langle u, q\rangle = 0 && \forall q \in \mathfrak{H}^k
\end{aligned}$$

Equivalently $\frac{1}{2}\langle \sigma, \sigma\rangle - \frac{1}{2}\langle du, du\rangle - \langle d\sigma, u\rangle - \langle u, p\rangle + \langle f, u\rangle \to \textit{saddle point}$

Special cases:

- $k = 0$: ordinary Laplacian
- $k = n$: mixed Laplacian
- $k = 1, n = 3$: $\sigma = -\operatorname{div} u$, $\operatorname{grad}\sigma + \operatorname{curl}\operatorname{curl} u = f$
- $k = 2, n = 3$: $\sigma = \operatorname{curl} u$, $\operatorname{curl}\sigma - \operatorname{grad}\operatorname{div} u = f$

Includes the problems

$$\operatorname{curl}\operatorname{curl} u = f,\ \operatorname{div} u = 0 \qquad \text{and} \qquad \operatorname{div} u = f,\ \operatorname{curl} u = 0$$

The mixed formulation is well-posed

Well-posedness of the mixed formulation follows directly from the Hodge decomposition and the Poincaré inequality. We must prove the inf-sup condition for the bilinear form

$$B(\sigma, u, p; \tau, v, q) = \langle \sigma, \tau\rangle - \langle d\tau, u\rangle - \langle d\sigma, v\rangle - \langle du, dv\rangle - \langle v, p\rangle - \langle u, q\rangle,$$

i.e., given $(\sigma, u, p) \in H\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$, we need to control $\|\sigma\|_{H\Lambda} + \|u\|_{H\Lambda} + \|p\|$ by a bounded choice of τ, v, and q.

$\tau = \sigma$ controls $\|\sigma\|$, $v = d\sigma$ controls $\|d\sigma\|$, $v = p$ controls $\|p\|$
$v = u$ controls $\|du\|$. *How to control* $\|u\|$*?*

Hodge decomp.: $u = d\eta + s + z$, $\eta \in H\Lambda^{k-1}, s \in \mathfrak{H}^k, z \in \mathfrak{Z}^{k\perp}$

$\tau = \eta$ controls $\|d\eta\|$ and $q = s$ controls $\|s\|$. *How to control* $\|z\|$*?*

Poincaré's inequality: $\|z\| \le c\|dz\| = c\|du\|$

Requirements for discretization

- Subcomplexes
- Bounded cochain projections

Discretization

Choose finite dimensional subspaces $\Lambda_h^{k-1} \subset H\Lambda^{k-1}$, $\Lambda_h^k \subset H\Lambda^k$.
Assume that $d\Lambda_h^{k-1} \subset \Lambda_h^k$ so we get a subcomplex.

$$\begin{tikzcd}
\cdots \arrow[r] & H\Lambda^{k-1} \arrow[r, "d^{k-1}"] & H\Lambda^k \arrow[r] & \cdots \\
\cdots \arrow[r] & \Lambda_h^{k-1} \arrow[r, "d^{k-1}"] \arrow[u, "\cup"'] & \Lambda_h^k \arrow[r] \arrow[u, "\cup"'] & \cdots
\end{tikzcd}$$

We may then define $\mathfrak{B}_h^k$, $\mathfrak{Z}_h^k$, $\mathfrak{H}_h^k = (\mathfrak{B}_h^k)^\perp \cap \mathfrak{Z}_h^k$

(and obtain the discrete Hodge decomp: $\Lambda_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus (\mathfrak{Z}_h^k)^\perp$)

Galerkin's method: discretize the mixed formulation with

$$\Lambda^{k-1},\ \Lambda^k,\ \mathfrak{H}^k \longrightarrow \Lambda_h^{k-1},\ \Lambda_h^k,\ \mathfrak{H}_h^k$$

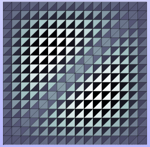
When is it stable, consistent, and convergent?

A simple case

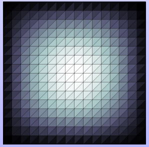
Stable discretization is not obvious, even in simple cases.

$\sigma \in H(div)$, $\quad u \in L^2$:

$$\langle \sigma, \tau \rangle - \langle \operatorname{div} \tau, u \rangle = 0 \qquad \forall \tau \in H(\operatorname{div})$$
$$\langle \operatorname{div} \sigma, v \rangle = \langle f, v \rangle \qquad \forall v \in L^2$$



$\mathcal{P}_1$-$\mathcal{P}_0$



Raviart–Thomas - $\mathcal{P}_0$

Bounded cochain projections

Key property: Suppose that there exists a bounded cochain projection.

$$\begin{tikzcd}
\cdots \arrow[r] & H\Lambda^{k-1} \arrow[r, "d^{k-1}"] \arrow[d, "\pi_h^{k-1}"] & H\Lambda^k \arrow[r] \arrow[d, "\pi_h^k"] & \cdots \\
\cdots \arrow[r] & \Lambda_h^{k-1} \arrow[r, "d^{k-1}"] & \Lambda_h^k \arrow[r] & \cdots
\end{tikzcd}$$

- π_h^k bounded
- π_h^k a projection
- $\pi_h^k d^{k-1} = d^{k-1} \pi_h^{k-1}$

Theorem

- *If $\|v - \pi_h^k v\| < \|v\|\ \forall v \in \mathfrak{H}^k$, then the induced map on cohomology is an isomorphism.*
- $\operatorname{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k) \le \sup\limits_{\substack{v \in \mathfrak{H}^k \\ \|v\|=1}} \|v - \pi_h^k v\|$
- *The discrete Poincaré inequality holds uniformly in h.*
- *Galerkin's method is stable and convergent.*

Proof of discrete Poincaré inequality

Thm. There is a positive constant c, independent of h, such that

$$\|\omega\| \le c\|d\omega\|, \quad \omega \in \mathfrak{Z}_h^{k\perp}.$$

Proof. Given $\omega \in \mathfrak{Z}_h^{k\perp}$, define $\eta \in \mathfrak{Z}^{k\perp} \subset H\Lambda^k(\Omega)$ by $d\eta = d\omega$. By the Poincaré inequality, $\|\eta\| \le c\|d\omega\|$, so it is enough to show that $\|\omega\| \le c\|\eta\|$. Now, $\omega - \pi_h\eta \in \Lambda_h^k$ and $d(\omega - \pi_h\eta) = 0$, so $\omega - \pi_h\eta \in \mathfrak{Z}_h^k$. Therefore

$$\|\omega\|^2 = \langle \omega, \pi_h\eta\rangle + \langle \omega, \omega - \pi_h\eta\rangle = \langle \omega, \pi_h\eta\rangle \le \|\omega\|\|\pi_h\eta\|,$$

whence $\|\omega\| \le \|\pi_h\eta\|$, and the result follows.

Subspaces of finite element differential forms

- Shape functions
- Degrees of freedom
- Dual bases and explicit bases
- Finite element de Rham subcomplexes
- Bounded cochain projections
- Applications

Finite element differential forms

Let $\mathcal{T} = \mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^n$. We wish to construct finite element spaces $\Lambda_h^k \subset H\Lambda^k(\Omega)$ which form a finite dimensional subcomplex with bounded cochain projections.

The Λ_h^k are finite element spaces in the sense that they can be *assembled* from the following data on each simplex:

- a finite dimensional space of polynomial forms on the simplex, and
- a decomposition of its dual space into subspaces associated to the subsimplices (degrees of freedom)

Special cases:

- Lagrange finite element spaces $\mathcal{P}_r\Lambda^0(\mathcal{T})$
- Discontinuous piecewise polynomials of degree r: $\mathcal{P}_r\Lambda^n(\mathcal{T})$.
- Whitney k-forms (one DOF per k-face): $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$

Construction of FE differential forms

Key to the construction is the Koszul differential $\kappa : \Lambda^k \to \Lambda^{k-1}$:

$$(\kappa\omega)_x(v^1, \dots, v^{k-1}) = \omega_x(x, v^1, \dots, v^{k-1})$$

$$0 \longleftarrow \mathcal{P}_r\Lambda^0 \xleftarrow{\kappa} \mathcal{P}_{r-1}\Lambda^1 \xleftarrow{\kappa} \cdots \xleftarrow{\kappa} \mathcal{P}_{r-n}\Lambda^n \longleftarrow 0$$

Koszul complex

$$0 \leftarrow \mathcal{P}_r\Lambda^0 \xleftarrow{\cdot X} \mathcal{P}_{r-1}\Lambda^1(\mathbb{R}^3) \xleftarrow{\times X} \mathcal{P}_{r-2}\Lambda^2(\mathbb{R}^3) \xleftarrow{X} \mathcal{P}_{r-3}\Lambda^3(\mathbb{R}^3) \leftarrow 0$$

C.f., the polynomial de Rham complex

$$0 \longrightarrow \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \longrightarrow 0$$

Key relation: $(d\kappa + \kappa d)\omega = (r+k)\omega \quad \forall \omega \in \mathcal{H}_r\Lambda^k$ (homogeneous polys)

κ is a contracting chain homotopy

Ex: $\operatorname{grad}(u \cdot x) + (\operatorname{curl} u)\times x = 3u$, $u \in \mathcal{H}_2(\mathbb{R}^3, \mathbb{R}^3)$

$$\boxed{\mathcal{H}_r\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}}$$

$\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$

Using the Koszul differential, we define a special space of polynomial differential k-forms between $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_{r-1}\Lambda^k$:

$$\mathcal{P}_r^-\Lambda^k := \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1} + \cancel{d\mathcal{H}_{r+1}\Lambda^{k-1}}$$

$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) = \begin{cases} \mathcal{P}_r\Lambda^k(\mathcal{T}_h), & k=0, \\ \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h), & k=n, \\ \text{strictly between}, & 0<k<n \end{cases}$$

$$\dim \mathcal{P}_r\Lambda^k = \binom{n+r}{n}\binom{n}{k} = \binom{n+r}{n-k}\binom{r+k}{k}$$
$$\dim \mathcal{P}_r^-\Lambda^k = \binom{n+r}{n-k}\binom{r+k-1}{k}$$

For each form degree k and polynomial degree r, $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ are *the* two natural finite element subspace of $H\Lambda^k(\Omega)$.

For example, they are invariant under pull-pack by any affine map $\mathbb{R}^n \to \mathbb{R}^n$, and they are almost the only such spaces of polynomial k-forms.

Degrees of freedom

To define the finite element spaces, we must specify *degrees of freedom*, i.e., a *decomposition of the dual spaces* $\mathcal{P}_r\Lambda^k(T)^*$ *and* $\mathcal{P}_r^-\Lambda^k(T)^*$, *into subspaces associated to subsimplices* f *of* T.

DOF for $\mathcal{P}_r^-\Lambda^k(T)$: to a subsimplex f of dim. $d \geq k$ we associate

$$\omega \mapsto \int_f \mathrm{Tr}_f\,\omega\wedge\eta, \quad \eta \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f) \qquad \text{Hiptmair}$$

DOF for $\mathcal{P}_r\Lambda^k(T)$: $\omega \mapsto \int_f \mathrm{Tr}_f\,\omega\wedge\eta, \quad \eta \in \mathcal{P}_{r+k-d}^-\Lambda^{d-k}(f)$

- The resulting FE spaces have precisely the continuity required by $H\Lambda^k$:

Theorem: $\mathcal{P}_r\Lambda^k(\mathcal{T}) = \{\, \omega \in H\Lambda^k(\Omega) \;:\; \omega|_T \in \mathcal{P}_r\Lambda^k(T) \quad \forall T \in \mathcal{T}\,\}$ *and similarly for* $\mathcal{P}_r^-$.

- The projections associated to these DOF commutes with d (Stokes theorem).

$$\begin{tikzcd} \Lambda^{k-1} \arrow[r, "d^{k-1}"] \arrow[d, "P_h^{k-1}"] & \Lambda^k \arrow[d, "P_h^k"] \\ \Lambda_h^{k-1} \arrow[r, "d^{k-1}"] & \Lambda_h^k \end{tikzcd}$$

Finite element differential forms/Mixed FEM

- $\mathcal{P}_r^-\Lambda^0(\mathcal{T}) = \mathcal{P}_r\Lambda^0(\mathcal{T}) \subset H^1$ Lagrange elts 
- $\mathcal{P}_r^-\Lambda^n(\mathcal{T}) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}) \subset L^2$ discontinuous elts 
- $n=2$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Raviart–Thomas elts 
- $n=2$: $\mathcal{P}_r\ \Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Brezzi–Douglas–Marini elts 
- $n=3$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 1st kind edge elts 
- $n=3$: $\mathcal{P}_r\ \Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 2nd kind edge elts 
- $n=3$: $\mathcal{P}_r^-\Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Nedelec 1st kind face elts 
- $n=3$: $\mathcal{P}_r\ \Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Nedelec 2nd kind face elts 

Ex: DOFs for $\mathcal{P}_3\Lambda^1$ on a tetrahedron

Cubic Nedelec 2nd kind edge elements, dim $= 60$

DOFs are:

- For each edge, moments versus $\mathcal{P}_3^-\Lambda^0(e)$ (4 DOF/edge $\times$ 6 edges)
- For each face, moments versus $\mathcal{P}_2^-\Lambda^1(f)$ (8 DOF/face $\times$ 4 faces)
- On the tet, moments versus $\mathcal{P}_1^-\Lambda^2(f)$ (4 DOF)

Dual bases

As a computational basis for $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ one choice is the dual basis to the degrees of freedom.

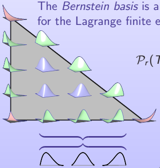
For $k = 0$ this is the standard Lagrange basis.

For $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ there is one basis element for each k-simplex $f = [x_{i_0}, \dots, x_{i_k}]$, namely the Whitney form given in barycentric coordinates by

$$\phi_{i_0\cdots i_k} := \sum_{p=0}^{k} (-1)^p \lambda_{i_p}\, d\lambda_{i_0}\wedge\cdots\wedge\widehat{d\lambda_{i_p}}\wedge\cdots\wedge d\lambda_{i_k}$$

$\mathcal{P}_1^-\Lambda^0 = \mathcal{P}_1\Lambda^0$:	λ_i
$\mathcal{P}_1^-\Lambda^1$:	$\lambda_i d\lambda_j - \lambda_j d\lambda_i$
$\mathcal{P}_1^-\Lambda^2$:	$\lambda_i\, d\lambda_j\wedge d\lambda_k - \lambda_j\, d\lambda_i\wedge d\lambda_k + \lambda_k\, d\lambda_i\wedge d\lambda_j$
etc.	

Explicit geometric bases



The *Bernstein basis* is an explicit alternative to the Lagrange basis for the Lagrange finite elts.

$$\mathcal{P}_r = \operatorname{span}\{\, \lambda^\alpha := \lambda_0^{\alpha_0}\cdots\lambda_n^{\alpha_n} \mid |\alpha| = r\,\}$$
$$\mathcal{P}_r(T,f) := \operatorname{span}\{\, \lambda^\alpha \mid \operatorname{spt}\alpha = \{i_0,\dots,i_k\},\ |\alpha| = r\,\}$$

$$\mathcal{P}_r(T) = \bigoplus_f \mathcal{P}_r(T,f)$$

$$\mathcal{P}_r(T,f) \xrightarrow[\mathrm{tr}]{\cong} \mathring{\mathcal{P}}_r(f) \cong \mathcal{P}_{r-\dim f-1}(f)$$

There are similar geometric bases for all k:

$$\mathcal{P}_r\Lambda^k(T) = \bigoplus_{\dim f\ge k} \mathcal{P}_r\Lambda^k(T,f),\ \mathcal{P}_r\Lambda^k(T,f) \xrightarrow[\mathrm{tr}]{\cong} \mathring{\mathcal{P}}_r\Lambda^k(f) \cong \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f-k}(f)$$

$$\mathcal{P}_r^-\Lambda^k(T) = \bigoplus_{\dim f\ge k} \mathcal{P}_r^-\Lambda^k(T,f), \mathcal{P}_r^-\Lambda^k(T,f) \xrightarrow[\mathrm{tr}]{\cong} \mathring{\mathcal{P}}_r^-\Lambda^k(f) \cong \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f-k}(f)$$

Example: explicit bases for $\mathcal{P}_r^-\Lambda^2$ and $\mathcal{P}_r\Lambda^2$ on a tet

$\mathcal{P}_r^-\Lambda^2$

r	Face $[x_i, x_j, x_k]$	Tet $[x_i, x_j, x_k, x_l]$
1	ϕ_{ijk}	
2	$\lambda_i\phi_{ijk},\ \lambda_j\phi_{ijk},\ \lambda_k\phi_{ijk}$	$\lambda_l\phi_{ijk},\ \lambda_k\phi_{ijl},\ \lambda_j\phi_{ikl}$

$\mathcal{P}_r\Lambda^2$

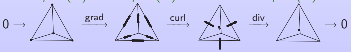
r	Face $[x_i, x_j, x_k]$	Tet $[x_i, x_j, x_k, x_l]$
1	$\lambda_k d\lambda_i\wedge d\lambda_j,\ \lambda_j d\lambda_i\wedge d\lambda_k,\ \lambda_i d\lambda_j\wedge d\lambda_k$	
2	$\lambda_k^2 d\lambda_i\wedge d\lambda_j,\ \lambda_j\lambda_k d\lambda_i\wedge d(\lambda_k-\lambda_j)$ $\lambda_j^2 d\lambda_i\wedge d\lambda_k,\ \lambda_i\lambda_j d(\lambda_j-\lambda_i)\wedge d\lambda_k$ $\lambda_i^2 d\lambda_j\wedge d\lambda_k,\ \lambda_i\lambda_k d\lambda_j\wedge d(\lambda_k-\lambda_i)$	$\lambda_k\lambda_l d\lambda_i\wedge d\lambda_j,\ \lambda_j\lambda_l d\lambda_i\wedge d\lambda_k$ $\lambda_j\lambda_k d\lambda_i\wedge d\lambda_l,\ \lambda_i\lambda_l d\lambda_j\wedge d\lambda_k$ $\lambda_i\lambda_k d\lambda_j\wedge d\lambda_l,\ \lambda_i\lambda_j d\lambda_k\wedge d\lambda_l$

Finite element de Rham subcomplexes

We don't only want spaces, we also want them to fit together into discrete de Rham complexes.

- One such FEdR subcomplex uses $\mathcal{P}_r^-\Lambda^k$ spaces of constant degree r:

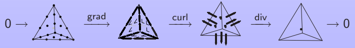
$$0 \to \mathcal{P}_r^-\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^-\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n(\mathcal{T}) \to 0$$



Whitney 1957, Bossavit 1988

- Another uses $\mathcal{P}_r\Lambda^k$ spaces with decreasing degree:

$$0 \to \mathcal{P}_r\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\mathcal{T}) \to 0$$



Demkowicz–Monk–Vardapetyan–Rachowicz 2000

- These are extreme cases. For every r $\exists\, 2^{n-1}$ such FEdR subcomplexes.

Constructing bounded cochain projections

We also need projections $\pi_h^k : H\Lambda^k \to \Lambda_h^k$ which are uniformly bounded in h and commute with d.

The projections P_h^k defined via the DOFs, commute with d. Unfortunately they are not bounded on $H\Lambda^k$.

So first we regularize, and then project: $Q_h^k u := P_h^k(\rho_\epsilon * u)$
This map is bounded, uniformly in h if $\epsilon = \delta h$, and still commutes with d. Unfortunately it is not a projection.

If we choose the δ sufficiently small, we can prove that $Q_h^k|_{\Lambda_h^k} : \Lambda_h^k \to \Lambda_h^k$ is close to the identity, uniformly in h, hence invertible. Then the composition

$$\pi_h^k = (Q_h^k|_{\Lambda_h^k})^{-1} \circ Q_h^k$$

is the desired *bounded cochain projection*.

Christiansen, Schöberl, AFW

Stable finite elements for the Hodge Laplacian

Putting this altogether we obtain four different stable families of mixed finite elements for the Hodge Laplacian.
(reduces to 2 for $k = n$ and 1 for $k = 0$)

$$\mathcal{P}_r^- \Lambda^{k-1}(\mathcal{T}) \times \mathcal{P}_r^- \Lambda^k(\mathcal{T})$$

$$\mathcal{P}_r \Lambda^{k-1}(\mathcal{T}) \times \mathcal{P}_r^- \Lambda^k(\mathcal{T})$$

$$\mathcal{P}_{r+1}^- \Lambda^{k-1}(\mathcal{T}) \times \mathcal{P}_r \Lambda^k(\mathcal{T})$$

$$\mathcal{P}_{r+1} \Lambda^{k-1}(\mathcal{T}) \times \mathcal{P}_r \Lambda^k(\mathcal{T})$$

For each we obtain optimal order estimates for σ and u.

Other applications

- Maxwell's equations and related EM problems
- Mixed eigenvalue problems
  (bounded cochain projections $\implies$ discrete compactness)
- Preconditioning and multigrid
- A posteriori error estimation
- Stable mixed FEM for elasticity

Application to elasticity

Stress–displacement mixed finite elements for elasticity

Find stress $\sigma : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}}$, displacement $u : \Omega \to \mathbb{R}^3$ such that

$$A\sigma = \epsilon(u), \quad \operatorname{div}\sigma = f$$

$$\int_\Omega \left(\frac{1}{2} A\,\sigma : \sigma + \operatorname{div}\sigma \cdot u + f \cdot u \right) \, dx \xrightarrow[H(\operatorname{div};\mathbb{S})\times L^2(\mathbb{R}^n)]{\sigma, u} \text{stationary point}$$

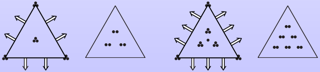
Search for stable finite elements dates back to the '60s, very limited success.

> "It is, of course, possible to derive elements that exhibit complete continuity of the appropriate components along interfaces and indeed this was achieved by Raviart and Thomas in the case of the heat conduction problem discussed previously. Extension to the full stress problem is difficult and as yet such elements have not been successfully noted."
>
> –Zienkiewicz, Taylor, Zhu
> *The Finite Element Method: Its Basis & Fundamentals*, 6th ed., 2005

Thanks to FEEC, we can retire that statement!

Recent progress coming from the FEEC perspective

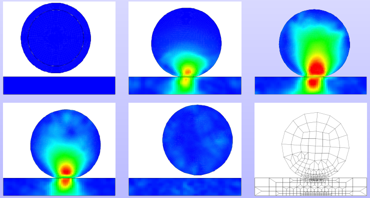
- First stable elements based on polynomials, 2D (Arnold–Winther 2002), all degrees $r \geq 1$:



- 3D stable elements, all degrees $r \geq 1$ (Arnold–Awanou–Winther 2007): for $r = 1$ stress space has 162 degrees of freedom (27 per component on average)

A computation using the new elements

From Eberhard, Hueber, Jiang, Wohlmuth 2006



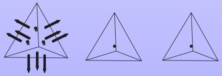
Mixed formulation with weak symmetry

Idea goes back to Fraeijs de Veubeke 1975, Amara–Thomas 1979
In the classical Hellinger–Reissner principle, symmetry of the stress tensor (balance of angular momentum) is assumed to hold exactly.
Instead we impose it weakly with a Lagrange multiplier (the rotation).

$$\int_\Omega \left(\frac{1}{2} A\,\sigma : \sigma + \operatorname{div}\sigma \cdot u + f \cdot u \right) \, dx \xrightarrow[H(\operatorname{div};\mathbb{S})\times L^2(\mathbb{R}^n)]{\sigma, u} \text{stationary point}$$

$$\int_\Omega \left(\frac{1}{2} A\,\sigma : \sigma + \operatorname{div}\sigma \cdot u + \sigma : p + f \cdot u \right) \, dx \xrightarrow[H(\operatorname{div};\mathbb{M})\times L^2(\mathbb{R}^n)\times L^2(\mathbb{K})]{\sigma, u, p} \text{S.P.}$$

FEEC has led to very simple stable elements



σ $\qquad$ u $\qquad$ p

The elasticity complex

There is a complex for elasticity analogous to the de Rham complex. It has versions both for strong symmetry and weak symmetry.

displacement rotation strain

$$0 \to H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{K}) \xrightarrow{(\mathrm{grad},-I)} H(J,\Omega;\mathbb{M}) \xrightarrow{J}$$
$$\xrightarrow{J} H(\mathrm{div},\Omega;\mathbb{M}) \xrightarrow{\begin{pmatrix}\mathrm{div}\\ \mathrm{skw}\end{pmatrix}} L^2(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{K}) \to 0$$

stress load couple

J is second order!

New mixed finite elements for elasticity

The elasticity complex can be derived from the de Rham complex by an intricate construction. Mimicking this construction on the discrete level we have derived stable mixed finite elements for elasticity. (Arnold-Falk-Winther 2006, 2007).

Main result

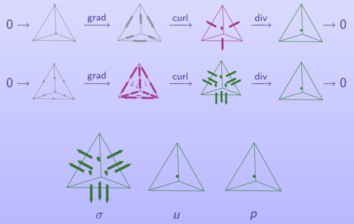
Choose *two* discretizations of the de Rham complex:

$$0 \longrightarrow \Lambda_h^0 \xrightarrow{\mathrm{grad}} \Lambda_h^1 \xrightarrow{\mathrm{curl}} \Lambda_h^2 \xrightarrow{\mathrm{div}} \Lambda_h^3 \longrightarrow 0$$
$$0 \longrightarrow \tilde\Lambda_h^0 \xrightarrow{\mathrm{grad}} \tilde\Lambda_h^1 \xrightarrow{\mathrm{curl}} \tilde\Lambda_h^2 \xrightarrow{\mathrm{div}} \tilde\Lambda_h^3 \longrightarrow 0$$

Surjectivity Hypothesis: (roughly) for each DOF of Λ_h^2 there is a corresponding DOF of $\tilde\Lambda_h^1$.

$$\text{Then} \left\{ \begin{array}{ll} \text{stress:} & \tilde\Lambda_h^2(\mathbb{R}^3) \\ \text{displacement:} & \tilde\Lambda_h^3(\mathbb{R}^3) \\ \text{rotation:} & \Lambda_h^3(\mathbb{K}) \end{array} \right\} \text{ is a stable element choice.}$$

The simplest choice



Features of the new mixed elements

- Based on HR formulation with weak symmetry; very natural
- Lowest degree element is very simple: full $\mathcal{P}_1$ for stress, $\mathcal{P}_0$ for displacement and rotation
- Works for every polynomial degree
- Works the same in 2 and 3 (or more) dimensions
- Robust to material constraints like incompressibility
- Provably stable and convergent

Conclusions

- Capturing the right geometric structure on the discrete level can be essential to get stable methods. For PDE related to the de Rham complex, this is achieved by using subspaces which form a subcomplex with bounded cochain projections.
- There are two natural families of finite element subspaces of $H\Lambda^k$, the spaces $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$. They can be assembled into complexes with bounded cochain projections (in numerous ways). They are inter-related and must be studied together.
- These spaces unify and clarify many known mixed finite element methods that were derived independently. This approach allows for a systematic way of choosing degrees of freedom, dual bases, and explicit geometric bases.
- Through FEEC we believe we have completed the long search for "just the right" mixed finite elements for elasticity.