Differential Complexes and Mixed Finite Elements for Elasticity

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Outline

1. Mixed finite elements for elasticity
2. Exterior calculus
3. Discrete exterior calculus
4. Construction of the elasticity complex

Coming Attractions!

Construction of the elements

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Linear elasticity

displacement $u : \Omega \to \mathbb{V} := \mathbb{R}^n$ $A\sigma = \epsilon\, u := [\nabla u + (\nabla u)^T]/2$

stress $\sigma : \Omega \to \mathbb{S} := \mathbb{R}^{n\times n}_{\text{sym}}$ $\operatorname{div}\sigma = f$

$\sigma \in H(\operatorname{div}, \Omega; \mathbb{S}), u \in L^2(\Omega; \mathbb{V})$ satisfy

$$\int_\Omega A\sigma : \tau\, dx + \int_\Omega \operatorname{div}\tau \cdot u\, dx = 0 \qquad \forall \tau \in H(\operatorname{div}, \Omega; \mathbb{S})$$

$$\int_\Omega \operatorname{div}\sigma \cdot v\, dx = \int_\Omega f \cdot v \qquad \forall v \in L^2(\Omega; \mathbb{V})$$

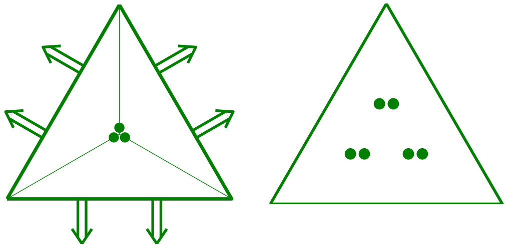
$(\sigma, u) \in H(\operatorname{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{V})$ saddle point of

$$\mathcal{L}(\tau, v) = \int_\Omega (\frac{1}{2} A\tau : \tau + \operatorname{div}\tau \cdot v - f \cdot v)\, dx.$$

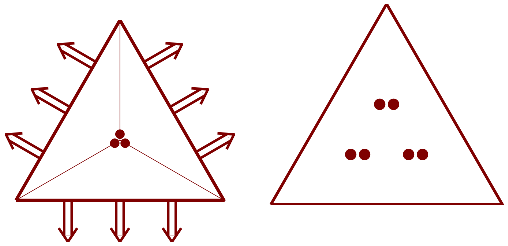
Stable mixed elements

2D composite elts: Johnson–Mercier ’78

cf. Fraeijs de Veubeke ’65; Watwood–Hartz ’68

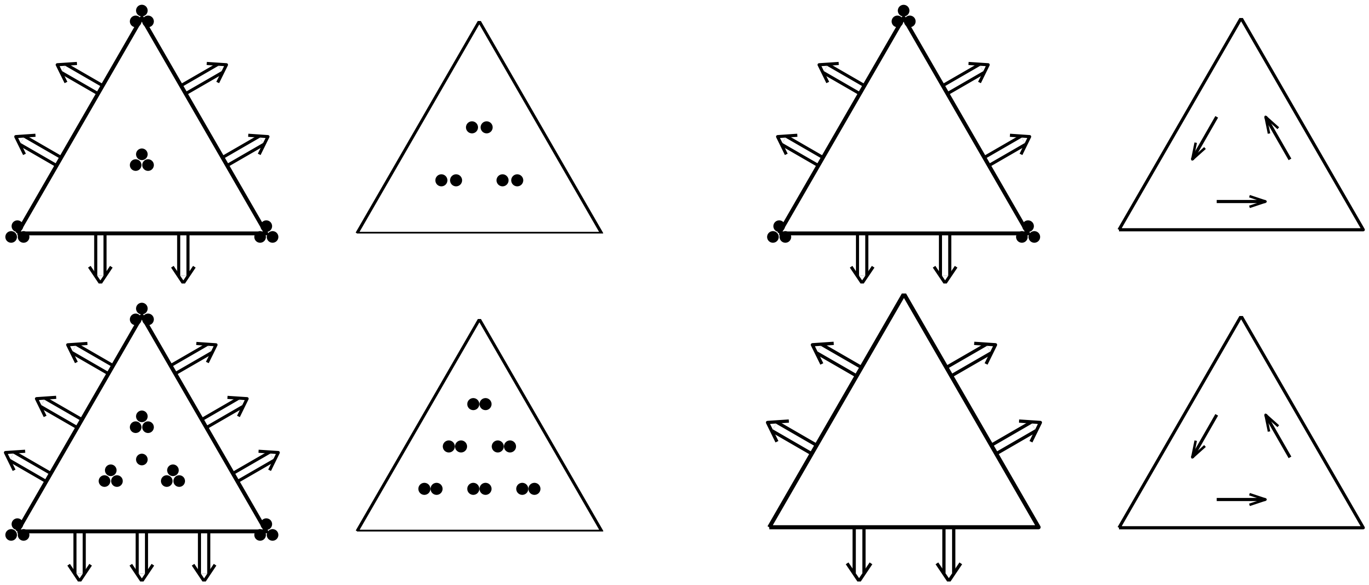


Arnold–Douglas–Gupta ’84



. . .

2D polynomial elements, Arnold–Winther 2002:



nonconf.

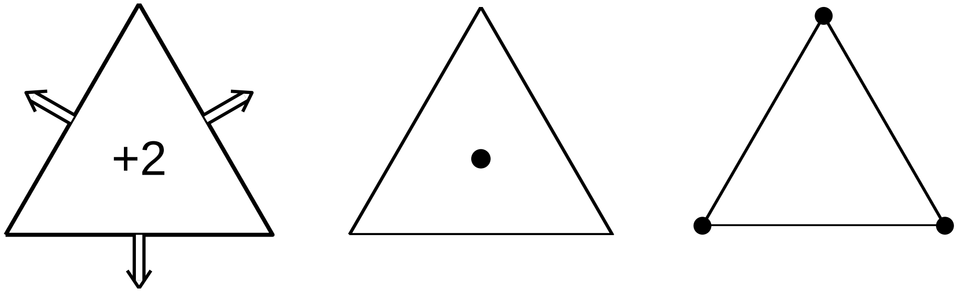
3D very complicated. . .

Weak symmetry

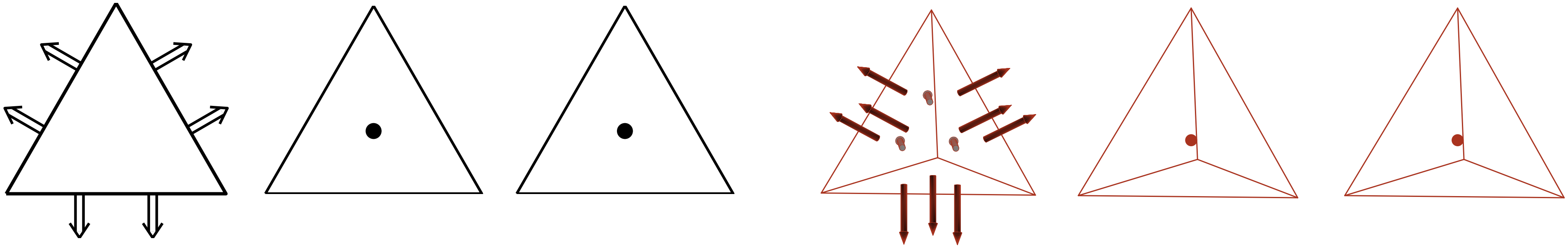
To avoid the problems arising from the symmetry of the stress tensor, it can be imposed *weakly*

$$(\sigma, u, p) = \operatorname*{argcrit}_{\substack{H(\mathrm{div},\Omega;\mathbb{M})\times L^2(\Omega;\mathbb{V})\\ \times L^2(\Omega;\mathbb{K})}} \left(\frac{1}{2}\int A\tau : \tau + \int v \cdot \operatorname{div}\tau + \int \tau : p + \int f \cdot v\right)$$

Fraejis de Veubeke '75, Arnold-Brezzi-Douglas '84 (PEERS), Stenberg, Morley, . . .



Major result of these talks: new elements in 2D and 3D



These were obtained via a new “homological” viewpoint. . .

The elasticity complexes

A key to developing stable elements for elasticity (with strongly imposed symmetry) is the *elasticity complex*:

$$\begin{tikzcd}
\mathbb{T} \arrow[r, hook] & C^\infty(\Omega,\mathbb{V}) \arrow[r, "\epsilon"] & C^\infty(\Omega,\mathbb{S}) \arrow[r, "J"] & C^\infty(\Omega,\mathbb{S}) \arrow[r, "\operatorname{div}"] & C^\infty(\Omega,\mathbb{V}) \arrow[r] & 0 \\
 & \text{displacement} \arrow[u] & \text{strain} \arrow[u] & \text{stress} \arrow[u] & \text{load} \arrow[u] &
\end{tikzcd}$$

$J = \operatorname{curl}_{\mathsf{c}} \operatorname{curl}_{\mathsf{r}}$, second order

$\mathbb{T}$ is the space of infinitesmal rigid motions

For weakly imposed symmetry the relevant sequence is

$$\begin{tikzcd}
\mathbb{T} \arrow[r, hook] & C^\infty(\mathbb{V}\times\mathbb{K}) \arrow[r, "{(\operatorname{grad}, -I)}"] & C^\infty(\mathbb{M}) \arrow[r, "J"] & C^\infty(\mathbb{M}) \arrow[r, "{\begin{pmatrix}\operatorname{div}\\ \operatorname{skw}\end{pmatrix}}"] & C^\infty(\mathbb{V}\times\mathbb{K}) \arrow[r] & 0
\end{tikzcd}$$

where J is defined to be zero on skew matrices.

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De Rham complex

de Rham complex:

$$0 \to \mathbb{R} \xrightarrow{\subset} \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \to 0$$

$\omega \in \Lambda^k(\Omega) \iff \omega_x$ is k-linear alternating form on $T_x\Omega\ \forall x \in \Omega$

L^2 de Rham complex:

$$0 \to \mathbb{R} \xrightarrow{\subset} H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0$$

$$H\Lambda^k(\Omega) = \{\, \omega \in L^2\Lambda^k(\Omega) \,|\, d\omega \in L^2\Lambda^{k+1}(\Omega) \,\}$$

Polynomial de Rham complexes

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{H}_r\Lambda^0 \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{H}_{r-n}\Lambda^n \to 0$$

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \to 0$$

Koszul complex

Koszul differential $\kappa : \Lambda^{k+1} \to \Lambda^k$:

$$(\kappa\omega)_x(v^1, \dots, v^k) = \omega_x(x, v^1, \dots, v^k)$$

- $\kappa : \mathcal{P}_r\Lambda^k \to \mathcal{P}_{r+1}\Lambda^{k-1}$ (c.f. $d : \mathcal{P}_{r+1}\Lambda^{k-1} \to \mathcal{P}_r\Lambda^k$)

$$0 \leftarrow \mathbb{R} \longleftarrow \mathcal{P}_r\Lambda^0 \xleftarrow{\kappa} \mathcal{P}_{r-1}\Lambda^1 \xleftarrow{\kappa} \cdots \xleftarrow{\kappa} \mathcal{P}_{r-n}\Lambda^n \leftarrow 0$$

Koszul complex

- $$(d\kappa + \kappa d)\omega = (r+k)\omega \quad \forall \omega \in \mathcal{H}_r\Lambda^k$$

κ is a contracting chain homotopy

- $$\mathcal{H}_r\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$$

$\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^+\Lambda^k$

Using the Koszul differential, we define a special space of polynomial differential k-forms between $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_{r+1}\Lambda^k$:

$$\mathcal{P}_r^+\Lambda^k := \mathcal{P}_r\Lambda^k + \kappa\mathcal{H}_r\Lambda^{k+1}$$

Note that $\mathcal{P}_r^+\Lambda^0 = \mathcal{P}_{r+1}\Lambda^0$ and $\mathcal{P}_r^+\Lambda^n = \mathcal{P}_r\Lambda^n$

* * *

God made $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^+\Lambda^k$,
all the rest is the work of man.

* * *

The case $\Omega \subset \mathbb{R}^3$

$$0 \to \mathbb{R} \xrightarrow{\subset} \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \Lambda^2(\Omega) \xrightarrow{d} \Lambda^3(\Omega) \to 0$$

$$0 \to \mathbb{R} \xrightarrow{\subset} C^\infty(\Omega) \xrightarrow{\mathrm{grad}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\mathrm{curl}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\mathrm{div}} C^\infty(\Omega) \to 0$$

smooth de Rham complex

$$0 \to \mathbb{R} \xrightarrow{\subset} H^1(\Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0$$

L^2 de Rham complex

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{P}_r(\Omega) \xrightarrow{\mathrm{grad}} \mathcal{P}_{r-1}(\Omega, \mathbb{R}^3) \xrightarrow{\mathrm{curl}} \mathcal{P}_{r-2}(\Omega, \mathbb{R}^3) \xrightarrow{\mathrm{div}} \mathcal{P}_{r-3}(\Omega) \to 0$$

polynomial de Rham complex

$$0 \leftarrow \mathbb{R} \longleftarrow \mathcal{P}_r(\Omega) \xleftarrow{\cdot x} \mathcal{P}_{r-1}(\Omega, \mathbb{R}^3) \xleftarrow{\times x} \mathcal{P}_{r-2}(\Omega, \mathbb{R}^3) \xleftarrow{x} \mathcal{P}_{r-3}(\Omega) \leftarrow 0$$

Koszul complex

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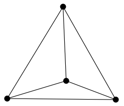
Piecewise polynomial differential forms

$\mathcal{T}$ a triangulation of $\Omega \subset \mathbb{V}$ by n-simplices

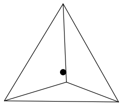
$$\mathcal{P}_r\Lambda^k(\mathcal{T}) := \{\omega \in H\Lambda^k(\Omega) \mid \omega|_T \in \mathcal{P}_r\Lambda^k(T) \quad \forall T \in \mathcal{T}\}$$

$$\mathcal{P}_r^+\Lambda^k(\mathcal{T}) := \{\omega \in H\Lambda^k(\Omega) \mid \omega|_T \in \mathcal{P}_r^+\Lambda^k(T) \quad \forall T \in \mathcal{T}\}$$

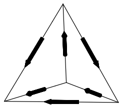
- $\mathcal{P}_r^+\Lambda^0(\mathcal{T}) = \mathcal{P}_{r+1}\Lambda^0(\mathcal{T}) \subset H^1$ Lagrange elts



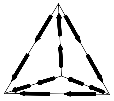
- $\mathcal{P}_r^+\Lambda^n(\mathcal{T}) = \mathcal{P}_r\Lambda^n(\mathcal{T}) \subset L^2$ discontinuous elts



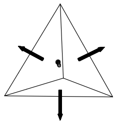
- $n = 3$: $\mathcal{P}_r^+\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 1st kind elts



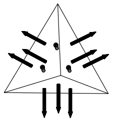
- $\mathcal{P}_r\,\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 2nd kind elts



- $\mathcal{P}_r^+\Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Raviart–Thomas elts



- $\mathcal{P}_r\,\Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Brezzi–Douglas–Marini elts



Degrees of freedom

T an n-simplex, $\Delta_d(T) =$ set of faces of dimension d, $0 \le d \le n$

DOF for $\mathcal{P}_r\Lambda^k(T)$:

$$u \mapsto \int_f u \wedge v, \quad v \in \mathcal{P}^+_{r-d-1+k}\Lambda^{d-k}(f), \quad f \in \Delta_d(T), \quad k \le d \le n$$

DOF for $\mathcal{P}^+_r\Lambda^k(T)$:

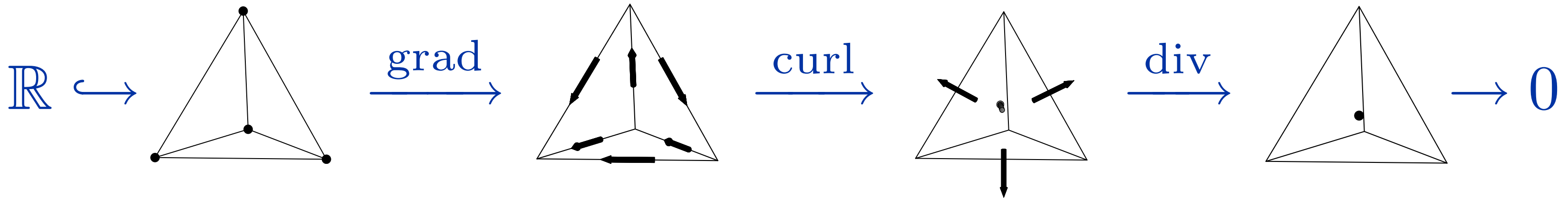
$$u \mapsto \int_f u \wedge v, \quad v \in \mathcal{P}_{r-d+k}\Lambda^{d-k}(f), \quad f \in \Delta_d(T_n), \quad k \le d \le n$$

Discrete exact sequences

For every $r \geq 0$, the $\mathcal{P}_r^+\Lambda^k$ spaces give an exact piecewise polynomial subcomplex:

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{P}_r^+\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^+\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^+\Lambda^n(\mathcal{T}) \to 0$$

For $n = 3$, $r = 0$ these are the Whitney elements:



For all r, the natural projections $\Pi_{r+}^k : \Lambda^k(\Omega) \to \mathcal{P}_r^+\Lambda^k(\mathcal{T})$ relate this to the de Rham sequence commutatively:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "\subset"] & \Lambda^0(\Omega) \arrow[r, "d"] \arrow[d, "\Pi_{r+}^0"'] & \Lambda^1(\Omega) \arrow[r, "d"] \arrow[d, "\Pi_{r+}^1"'] & \cdots \arrow[r, "d"] & \Lambda^n(\Omega) \arrow[r] \arrow[d, "\Pi_{r+}^n"'] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\subset"] & \mathcal{P}_r^+\Lambda^0(\mathcal{T}) \arrow[r, "d"] & \mathcal{P}_r^+\Lambda^1(\mathcal{T}) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \mathcal{P}_r^+\Lambda^n(\mathcal{T}) \arrow[r] & 0
\end{tikzcd}$$

Other discrete exact sequences

Another exact sequence ending at $\mathcal{P}_r\Lambda^n(\mathcal{T})$ uses the $\mathcal{P}_s\Lambda^k$ spaces of increasing degree: (Demkowicz-Vardepetyan '99):

$$\mathbb{R} \xrightarrow{\subset} \mathcal{P}_{r+n}\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r+n-1}\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r\Lambda^n(\mathcal{T}) \to 0$$

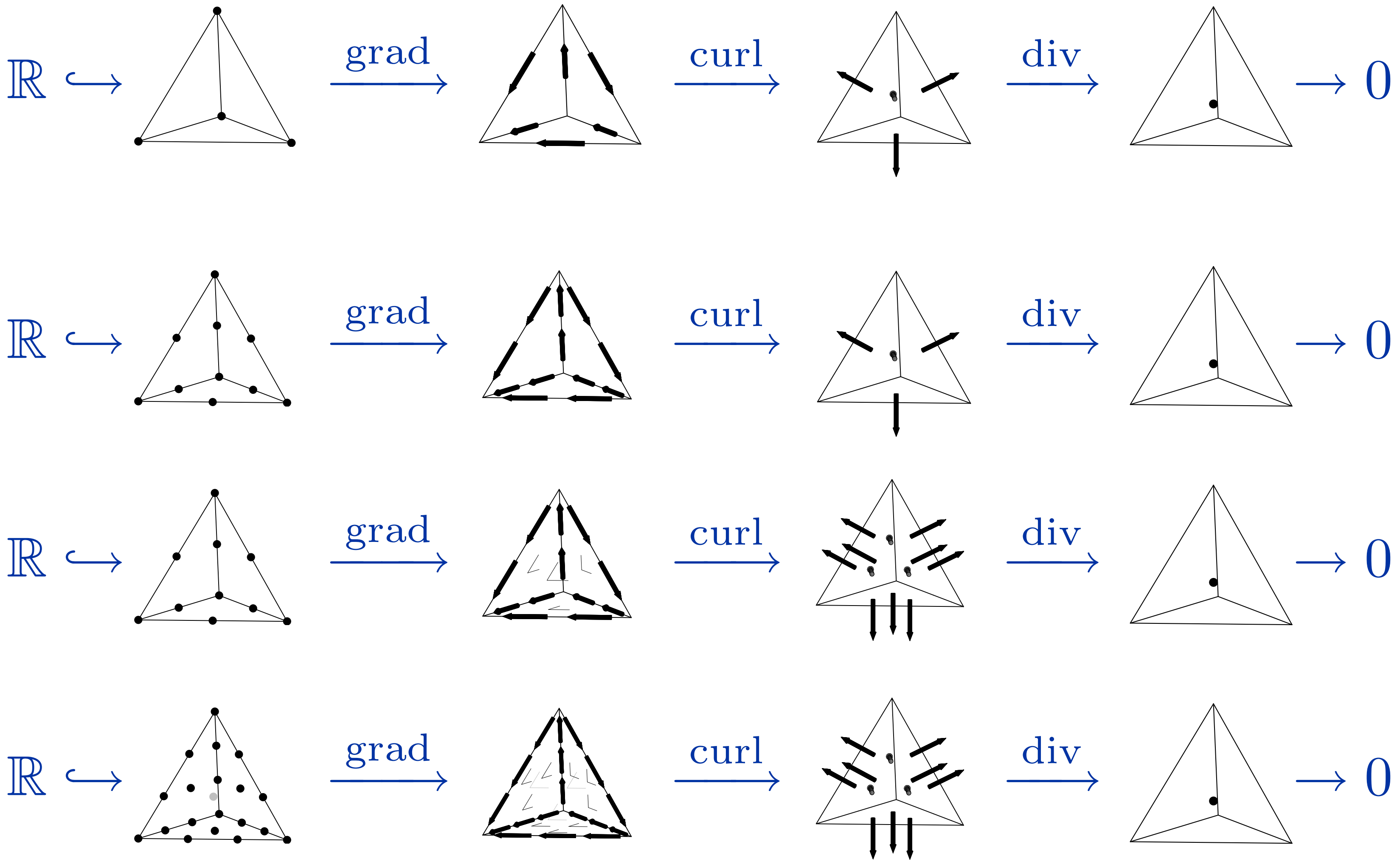
In fact, these are just the extreme cases. There are 2^{n-1} pw polynomial de Rham sequences in n dimensions. All relate to the de Rham complex through a commuting diagram.

For $n = 3$ the other two are:

$$\mathbb{R} \xrightarrow{\subset} \mathcal{P}_{r+2}\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^1(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^+\Lambda^2(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r\Lambda^3(\mathcal{T}) \to 0$$

$$\mathbb{R} \xrightarrow{\subset} \mathcal{P}_{r+2}\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r+1}^+\Lambda^1(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^2(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r\Lambda^3(\mathcal{T}) \to 0$$

The four sequences ending with $\mathcal{P}_0\Lambda^3(\mathcal{T})$ in 3D



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Bernstein–Gelfand–Gelfand construction, I

1. Start with the de Rham sequence with values in $\mathbb{W} := \mathbb{K} \times \mathbb{V}$:

$$\mathbb{W} \hookrightarrow \Lambda^0(\Omega;\mathbb{W}) \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)} \Lambda^1(\Omega;\mathbb{W}) \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)} \Lambda^2(\Omega;\mathbb{W}) \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)} \Lambda^3(\Omega;\mathbb{W}) \longrightarrow 0$$

2. For any $x \in \mathbb{R}^3$ define $K_x : \mathbb{V} \to \mathbb{K}$ by $K_x v = 2\,\mathrm{skw}(xv^T)$ and $K : \Lambda^k(\Omega;\mathbb{V}) \to \Lambda^k(\Omega;\mathbb{K})$ by

$$(K\omega)_x(v_1,\ldots,v_k) = K_x[\omega_x(v_1,\ldots,v_k)].$$

3. Define automorphisms $\Phi : \Lambda^k(\mathbb{W}) \to \Lambda^k(\mathbb{W})$ by

$$\Phi = \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}, \qquad \Phi^{-1} = \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}$$

4. Define $\mathcal{A} = \Phi \circ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \circ \Phi^{-1}$ to get a modified de Rham sequence:

$$\Phi(\mathbb{W}) \hookrightarrow \Lambda^0(\mathbb{W}) \xrightarrow{\mathcal{A}} \Lambda^1(\mathbb{W}) \xrightarrow{\mathcal{A}} \Lambda^2(\mathbb{W}) \xrightarrow{\mathcal{A}} \Lambda^3(\mathbb{W}) \to 0$$

Bernstein–Gelfand–Gelfand construction, II

5. Note that $\mathcal{A} = \begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}$, where $S = dK - Kd : \Lambda^k(\Omega; \mathbb{V}) \to \Lambda^{k+1}(\Omega; \mathbb{K})$ is given by

$$(S\omega)_x(v_1, \dots, v_{k+1}) = \sum_\mu \operatorname{sign}(\mu) K_{v_{\mu_{k+1}}} \omega_x(v_{\mu_1}, \dots, v_{\mu_k}).$$

Properties: S is algebraic; for $k = 1$, S is an isomorphism; $dS = -Sd$
$d(dK - Kd) = -dKd = -(dK - Kd)d$

6. Define subspaces $\Gamma^k \subset \Lambda^k(\Omega; \mathbb{W})$ satisfying $\mathcal{A}(\Gamma^k) \subset \Gamma^{k+1}$ and projections

$$\pi_k : \Lambda^k(\Omega; \mathbb{W}) \to \Gamma^k \qquad \text{satisfying} \qquad \pi_{k+1}\mathcal{A} = \mathcal{A}\pi_k :$$

$$\Gamma^0 = \Lambda^0(\Omega; \mathbb{W}), \quad \pi_0 = id, \quad \Gamma^3 = \Lambda^3(\Omega; \mathbb{W}), \quad \pi_3 = id,$$

$$\Gamma^1 = \{\, (\omega, \mu) \in \Lambda^1(\Omega; \mathbb{W}) : d\omega = S\mu \,\}, \quad \Gamma^2 = \{\, (\omega, \mu) \in \Lambda^2(\Omega; \mathbb{W}) : \omega = 0 \,\}$$

$$\pi^1 = \begin{pmatrix} I & 0 \\ S^{-1}d & 0 \end{pmatrix} : \Lambda^1(\Omega; \mathbb{W}) \to \Gamma^1, \quad \pi^2 = \begin{pmatrix} 0 & 0 \\ dS^{-1} & I \end{pmatrix} : \Lambda^2(\Omega; \mathbb{W}) \to \Gamma^2.$$

Bernstein–Gelfand–Gelfand construction, III

6. The following diagram commutes (use $dS = -Sd$):

$$\begin{tikzcd}
\Phi(\mathbb{W}) \arrow[r, hook] & \Lambda^0(\mathbb{W}) \arrow[r, "\mathcal{A}"] \arrow[d, "\pi_0"] & \Lambda^1(\mathbb{W}) \arrow[r, "\mathcal{A}"] \arrow[d, "\pi^1"] & \Lambda^2(\mathbb{W}) \arrow[r, "\mathcal{A}"] \arrow[d, "\pi^2"] & \Lambda^3(\mathbb{W}) \arrow[r] \arrow[d, "\pi_3"] & 0 \\
\Phi(\mathbb{W}) \arrow[r, hook] & \Gamma^0 \arrow[r, "\mathcal{A}"] & \Gamma^1 \arrow[r, "\mathcal{A}"] & \Gamma^2 \arrow[r, "\mathcal{A}"] & \Gamma^3 \arrow[r] & 0
\end{tikzcd}$$

Therefore, the subcomplex on the bottom row is exact.

7. This subcomplex may be identified with the elasticity complex.

Bernstein–Gelfand–Gelfand construction, concluded

$$\begin{tikzcd}
\Gamma^0 \arrow[r, "\mathcal{A}"] \arrow[d, phantom, "="] & \Gamma^1 \arrow[r, "\mathcal{A}"] \arrow[d, phantom, "\cong"] & \Gamma^2 \arrow[r, "\mathcal{A}"] \arrow[d, phantom, "\cong"] & \Gamma^3 \arrow[d, phantom, "="] \\
\Lambda^0(\mathbb{K}\times\mathbb{V}) \arrow[r, "{(d_0,-S_0)}"] & \Lambda^1(\Omega;\mathbb{K}) \arrow[r, "d_1\circ S_1^{-1}\circ d_1"] & \Lambda^2(\Omega;\mathbb{V}) \arrow[r, "{(-S_2,d_2)^T}"] & \Lambda^3(\mathbb{K}\times\mathbb{V})
\end{tikzcd}$$

With the identifications

$$\Lambda^0(\mathbb{K}\times\mathbb{V}) \leftrightarrow C^\infty(\mathbb{V}\times\mathbb{K})$$

$$\Lambda^1(\mathbb{K}) \leftrightarrow C^\infty(\mathbb{M})$$

$$\Lambda^2(\mathbb{K}) \leftrightarrow C^\infty(M)$$

$$\Lambda^3(\mathbb{K}\times\mathbb{V}) \leftrightarrow C^\infty(\mathbb{V}\times\mathbb{K})$$

this becomes the elasticity sequence

$$\mathcal{T}' \hookrightarrow C^\infty(\mathbb{V}\times\mathbb{K}) \xrightarrow{(\mathrm{grad},-I)} C^\infty(\mathbb{M}) \xrightarrow{J} C^\infty(\mathbb{M}) \xrightarrow{(\mathrm{div},\mathrm{skw})^T} C^\infty(\mathbb{V}\times\mathbb{K}) \to 0$$