Two ways to use our knowledge of 1-var. calculus to compute/define derivatives of multi-variable functions like $f: \mathcal{U} \to \mathbb{R}^m$, $\mathcal{U}$ open in $\mathbb{R}^n$

(1) Try to generalize one-variable difference quotient (in class Mon.)

(2) Treat our functions on $\mathbb{R}^n$ as depending on only one component "partial derivatives" (in section)

So derivative at point $\underline{a}$ in $x_i$-direction:

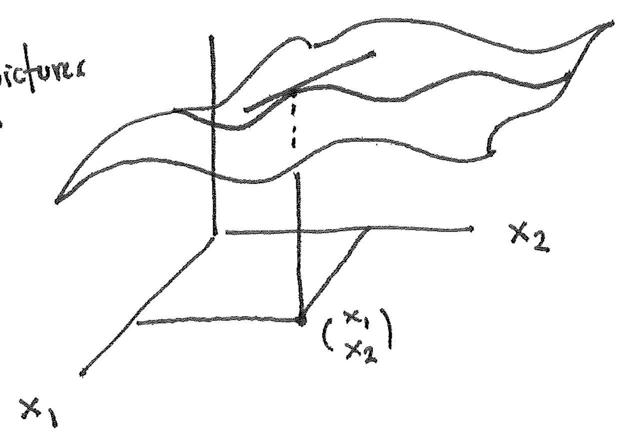
$$D_i f(\underline{a}) = \lim_{h \to 0} \frac{f\begin{pmatrix} a_1 \\ \vdots \\ a_i + h \\ \vdots \\ a_n \end{pmatrix} - f\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}{h}$$

and if our function is differentiable at all points in $\mathcal{U}$, replace $\underline{a}$ with variable $\underline{x}$, treat derivative as a function of $\underline{x}$.

<u>Ex</u>: $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \cos x_1 \\ x_2 \\ x_1 x_2^2 \end{bmatrix}$

then $D_1 f = \begin{bmatrix} -\sin x_1 \\ 0 \\ x_2^2 \end{bmatrix}$, $D_2 f = \begin{bmatrix} 0 \\ 1 \\ 2x_1 x_2 \end{bmatrix}$

In pictures




$D_2 f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is "slope" of tangent line in direction of ~~$x_2$~~ $\vec{e}_2$ as a vector in $\mathbb{R}^m$.

(our picture is $\mathbb{R}^2 \to \mathbb{R}$ so a 1-dim'l vector) in general, for $f : \mathbb{R}^n \to \mathbb{R}^m$, there are $m \times n$ partial derivatives to take

since $f(\underline{x}) = \begin{bmatrix} f_1(\underline{x}) \\ \vdots \\ f_m(\underline{x}) \end{bmatrix}$ and $\underline{x} = (x_1, \dots, x_n)$.

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At the end of Monday's class we noticed that we could define $f'(a)$ as the unique real number s.t.

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - h \cdot f'(a)}{|h|} = 0.$$

key pt. being that we could replace $h$ with $|h|$, and this definition good for generalizations.

Why is it unique? If another real # $r$ also gave limit 0, then subtracting two expressions (limit is well-behaved under subtraction)

$$\lim_{h \to 0} \frac{h(r - f'(a))}{|h|} = 0 \Rightarrow r - f'(a) = 0.$$

since $h/|h| \to \pm 1$ as $h \to 0^{\pm}$

Plan: Define derivative as unique map (?) $\overset{Df(\underline{a})}{\vee}$ from $\mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\underline{h} \to \underline{0}} \frac{f(\underline{a}+\underline{h}) - f(\underline{a}) - Df(\underline{a})(h)}{|h|} = 0.$$

KEY POINT: $Df(\underline{a})$ is linear! Just as we view $f'(a)$ as linear transformation $\mathbb{R}^1 \to \mathbb{R}^1$

$x \mapsto f'(a)x$

or better: $h \mapsto f'(a)h$

Proposition: ① if $Df(\underline{a})$ exists, it is unique

pf: Same as for linear map $\mathbb{R}^1 \to \mathbb{R}^1$ — we consider another linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ for which the assoc. limit $= \underline{0}$, the 0-vector in $\mathbb{R}^m$

then subtracting two limits:

$$\lim_{\underline{h} \to \underline{0}} \frac{(Df(\underline{a}) - L)(\underline{h})}{|\underline{h}|} = \underline{0}. \quad (*)$$

$Df(\underline{a})$, $L$ will be same if their action on standard basis $\vec{e}_i$, $i = 1, \dots, n$ is same. Pick path $\underline{h} = t \cdot \vec{e}_i$ ($t \in \mathbb{R}$ scalar)

(for limit to exist, must exist in this direction $t \cdot \vec{e}_i$.)

then $(*)$ becomes

$$\lim_{t \to 0} \frac{(Df(\underline{a}) - L)(t\vec{e}_i)}{|t|} = \underline{0}$$

pull out scalar $t$, so get

$$(Df(\underline{a}) - L)(\vec{e}_i) = 0.$$

✓.

② if $Df(\underline{a})$ exists then matrix for $Df(\underline{a})$ is the Jacobian with entries $a_{ij} = \frac{\partial f^{(i)}}{\partial x_j}(\underline{a})$.

pf: as above, pick paths $\underline{h} = t\vec{e}_j$.

then $(*)$: $$\lim_{t \to 0} \frac{f(\underline{a} + t\vec{e}_j) - f(\underline{a}) - Df(\underline{a})(t\vec{e}_j)}{|t|} = \underline{0}$$ but can be rewritten as.

$$= \begin{cases} \left(\frac{\partial f}{\partial x_j}(\underline{a}) - Df(\underline{a})(\vec{e}_j)\right) & \text{if } t > 0 \\ -\left( \text{——}"\text{——} \right) & \text{if } t < 0 \end{cases} \quad . \checkmark$$

$\frac{\partial f}{\partial x_j}(\underline{a})$ means $\begin{bmatrix} \frac{\partial f^{(1)}}{\partial x_j}(\underline{a}) \\ \vdots \\ \frac{\partial f^{(m)}}{\partial x_j}(\underline{a}) \end{bmatrix}$

calculate tangent plane for

$$f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 y \quad \text{at} \quad \underline{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$z = f\begin{bmatrix} 3 \\ 1 \end{bmatrix} + Df\begin{bmatrix} 3 \\ 1 \end{bmatrix}\begin{bmatrix} x-3 \\ y-1 \end{bmatrix} = 9 + 6(x-3) + 9\cdot(y-1)$$

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]\Bigg|_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} = \left[2xy, x^2\right]\Bigg|_{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} = [6, 9].$$

WEDNESDAY LECTURE ENDED HERE

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How do we know when function is not differentiable?

If $f$ differentiable, then $f$ continuous. (numerator in limit must go to $\vec{0}$ for limit to exist.

$$\lim_{\underline{h}\to 0} \frac{f(\underline{a}+\underline{h}) - f(\underline{a}) - Df(\underline{a})\underline{h}}{|h|} \cdot |h|$$

)

so if $f$ not continuous, then $f$ not diff.

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But $\lim_{\underline{h}\to 0} Df(\underline{a})h = \underline{0}$.

Harder example: $f: \mathbb{R}^2 \to \mathbb{R}$. defined by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x^2 y}{x^2+y^2} \quad (x,y) \neq (0,0), \quad f(\underline{0}) = 0$$

numerator going to 0 faster than denom, so will be continuous at $\underline{0}$.

$Df(\underline{0})$ = Jacobian at $(0,0)$ if it exists.

compute it. See if it is 0-matrix.

$f\begin{pmatrix} x \\ 0 \end{pmatrix} = 0 \quad \forall\, x, y.$

$f\begin{pmatrix} 0 \\ y \end{pmatrix} = 0$

So difference quotient reduces to

$$\lim_{\underline{h}\to 0} \frac{f(\underline{h})}{|\underline{h}|} = \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^2 h_2}{(h_1^2+h_2^2)^{3/2}}$$

pick $h_1 = h_2$.