What can go wrong when attempting to differentiate a function $f: \mathcal{U} \to \mathbb{R}^m$ with $\mathcal{U}$ open $\subseteq \mathbb{R}^n$. ?

<u>Problem</u> 1 : f can fail to be continuous at $\underline{a}$. Intuition: tangent hyperplane shouldn't be a good approx. at $\underline{x} = \underline{a}$.

In 1-variable calculus, we prove that

f diff $\Rightarrow$ f continuous (i.e. f not continuous $\Rightarrow$ f not differentiable)

(contrapositive)

Similar pf. works in multi-var. setting though as usual, working with limit involving $Df(\underline{a})$, not just difference quotient with $f(\underline{a}+\underline{h}) - f(\underline{a})$, requires more care.

pf

Want to show : $\lim_{\underline{h}\to\underline{0}} f(\underline{a}+\underline{h}) - f(\underline{a}) = \underline{0}$.

Know by assumption that $\lim_{\underline{h}\to\underline{0}} \dfrac{f(\underline{a}+\underline{h}) - f(\underline{a}) - Df(\underline{a})(\underline{h})}{|h|} = \underline{0}$.

But then $\lim_{\underline{h}\to 0} |h| \cdot \left(\text{—"—} \right) = \underline{0}$. So

$$\lim_{\underline{h}\to\underline{0}} f(\underline{a}+\underline{h}) - f(\underline{a}) = \underbrace{\lim_{\underline{h}\to 0} \frac{|\underline{h}|}{|\underline{h}|}\left(f(\underline{a}+\underline{h}) - f(\underline{a}) - Df(\underline{a})(\underline{h}) \right)}_{= \underline{0} \text{ since } f \text{ diff.}} + \underbrace{\lim_{\underline{h}\to 0} \frac{Df(a)(\underline{h}) \cdot |\underline{h}|}{|\underline{h}|}}_{= \underline{0}}$$

since both these limits exist

$= \underline{0}$ since it is the value of a linear transformation on $\underline{h}$ as $\underline{h} \to 0$.

So f continuous.

<u>Q</u> : If partial derivatives (i.e. directional derivatives) in all directions exist, is the function differentiable?

A : No. Previous examples where limit of a function was $\underline{0}$ along any line, but when we approached along $y = x^2$, got different answer.

(e.g. $f\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{|y|\, e^{-|y|/x^2}}{x^2}$)

had limit along $y = x^2$ equal to e^{-1}.

Another example : $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is continuous but not differentiable at $(0,0)$.

$Df(\underline{0}) = 0\text{-matrix}$ since $f\begin{pmatrix} x \\ 0 \end{pmatrix} = f\begin{pmatrix} 0 \\ y \end{pmatrix} = \underline{0} \ \forall\ x, y$.

so investigating difference quotient reduces to

$$\lim_{\underline{h} \to \underline{0}} \frac{f(\underline{h})}{|\underline{h}|} = \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^2 h_2}{(h_1^2+h_2^2)^{3/2}}$$

Pick path $h_1 = h_2$.
Approach as this $h_1 > 0$, $h_1 < 0$.

~~Final example : $\sin \frac{1}{x}$. As $x \to 0$, this is oscillating faster and faster between -1 and 1.~~

~~But we can dampen these oscillations by multiplying by function $\to 0$ as $x \to 0$ like x^n. Try it, you'll see if $n \geq 2$ then differentiable at origin:~~

SKIP THIS!

$$f'(0) = \lim_{h \to 0} \frac{1}{h}\big(f(h) - \underbrace{f(0)}_{=0 \ \forall\, n \geq 1} \big) = \lim_{h\to 0} h^{n-1} \cdot \sin\frac{1}{h} = \begin{cases} 0 & \text{if } n \geq 2 \\ \text{no limit} & \text{if } n = 1. \end{cases}$$

~~But tangent line $y = 0$ isn't good approximation, since f is both increasing and decreasing in any neighborhood (i.e. open ball) around origin.~~

So we're lead to study functions with <u>continuous</u> first partial derivatives.

(remember if all partial derivs exist, not nec. true that all directional derivatives exist. But we'll see that continuity guarantees this.)

<u>New notation</u>: $f \in C^1(\mathcal{U})$: continuous first partials exist on $\mathcal{U}$.

(similarly $C^p(\mathcal{U})$: can differentiate p times and each is continuous.)
(take partials)

and hierachy $C^0(\mathcal{U}) \supset D^1(\mathcal{U}) \supset C^1(\mathcal{U}) \supset D^2(\mathcal{U}) \cdots$

<u>Big theorem</u>: If $\mathcal{U}$ open in $\mathbb{R}^n$, $f : \mathcal{U} \to \mathbb{R}^m$ in $C^1(\mathcal{U})$, then f is differentiable on $\mathcal{U}$ (with derivative given by Jacobian, as usual)

<u>pf</u>: We must show, with these assumptions,

$$\lim_{\underline{h} \to \underline{0}} \frac{1}{|\underline{h}|} \left(f(\underline{a}+\underline{h}) - f(\underline{a}) - \underbrace{Df(\underline{a})}_{\substack{\text{linear trans.} \\ \text{given by Jacobian} \\ \text{matrix.}}} h \right) = \underline{0}. \quad (*)$$

We may assume $m=1$, so

$f : \mathcal{U} \to \mathbb{R}$ since we can work component by component for each $f^{(i)}$, $i=1,\dots,m$

<u>Plan</u> - rewrite $f(\underline{a}+\underline{h}) - f(\underline{a})$ as differences of partial derivatives numerators:

$$f\begin{pmatrix} a_1+h_1 \\ \vdots \\ a_n+h_n \end{pmatrix} - f(\underline{a}) = f\begin{pmatrix} a_1+h_1 \\ \vdots \\ a_n+h_n \end{pmatrix} - f\begin{pmatrix} a_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix}$$
$$+ f\begin{pmatrix} a_1 \\ a_2+h_2 \\ \vdots \\ a_n+h_n \end{pmatrix} - f\begin{pmatrix} a_1 \\ a_2 \\ a_3+h_3 \\ \vdots \\ a_n+h_n \end{pmatrix}$$
$$\cdots$$

now we use MVT for multivariable functions to argue that each of these differences is achieved by $h_i \cdot D_i f(\underline{c_i})$ some c_i on line between two points.

think of them as functions of 1 var! so MVT applies

sum to 0 as do all others in list.

So the left-hand side of (*) is, for some $\underline{c}_i$, $i=1,\dots,n$,

$$= \lim_{\underline{h}\to 0} \frac{1}{|\underline{h}|}\left[\sum_{i=1}^{n} h_i \cdot D_i f(\underline{c}) - \underbrace{Df(\underline{a})(\underline{h})}_{\text{expand matrix mult.}}\right]$$

$$\begin{pmatrix} | & & | \\ D_1 f(\underline{a}) & \cdots & D_n f(\underline{a}) \\ | & & | \end{pmatrix}(\underline{h}) = \sum_{i=1}^{n} h_i D_i f(\underline{a})$$

$$= \lim_{\underline{h}\to 0} \frac{1}{|\underline{h}|} \sum_{i=1}^{n} h_i \left(D_i f(\underline{c}) - D_i f(\underline{a})\right)$$

Show that $|\cdot| \to 0$ as $\underline{h}\to 0$.

then we have $\displaystyle\lim_{\underline{h}\to 0} \frac{1}{|\underline{h}|}\left|\sum_{i=1}^{n} h_i \left(D_i f(\underline{c}) - D_i f(\underline{a})\right)\right|$

$$\le \lim_{\underline{h}\to 0} \sum_{i=1}^{n} \underbrace{\frac{|h_i|}{|\underline{h}|}}_{\le 1} \left(\underbrace{D_i f(\underline{c})}_{\to 0 \text{ as } \underline{h}\to 0 \text{ for all } i \text{ by continuity of first partials}} - D_i f(\underline{a})\right) = \underline{0}. \quad \checkmark$$