On to chapter 2 - solving equations. (Good time to remind you: READ THE BOOK!)

Talked a lot about subspaces in chapter 1.

Can represent them as span of vectors : pick $\vec{v}_1, \dots, \vec{v}_k$

consider all linear combinations $\{ c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_i \in \mathbb{R} \}$

Another way to represent subspace :

set $\{ \vec{v} \mid A \cdot \vec{v} = \underline{0} \}$ A: linear transformation ($m \times n$ matrix)

(check that solutions are well behaved under addition, scalar mult. Or think of $A = \begin{bmatrix} | & & | \\ A(\vec{e}_1) & \cdots & A(\vec{e}_n) \\ | & & | \end{bmatrix}$

then in review for exam we showed

$\{ \vec{v} \mid A(\vec{e}_i) \cdot \vec{v} = 0 \}$ is a subspace.

the above set is the intersections of these for $i = 1, \dots, n$.

<u>Goal</u> : Given ways of moving from one description to the other.

Key tool is row reduction / Gaussian elimination.

Solves more generally equations $A \cdot \vec{x} = \vec{b}$, $\vec{b} \in \mathbb{R}^m$.

Written out $A \cdot \vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots \\ \\ \\ \end{bmatrix}$

Want it $= \vec{b} = \begin{bmatrix} b_1 \\ \vdots \end{bmatrix}$ so must agree at each component

so get system of equations

$a_{11}x_1 + a_{12}x_2 + \dots = b_1$

$\vdots$

Give an algorithm for solving such a system.

these methods will allow us to

- solve linear / non-linear equations ↖ approx. the latter with tangent hyperplanes
  do multi-variable Newton's method.
- develop a more precise language for subspaces (dimension, etc.)
  to say when solutions exist, and if so, how many?
- in Ch. 3, allows us to solve optimization problems in multiple variables.

Given system of equations, like

$$\begin{aligned} x_1 - x_2 &= 4 \\ x_1 + 2x_2 &= 1 \end{aligned} \quad \Longleftrightarrow \quad \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

subtract top from bottom:

$3x_2 = -3$ so $x_2 = -1$

(if being more careful to record steps:

original system $\Longleftrightarrow$ system $\begin{aligned} x_1 - x_2 &= 4 \\ 3x_2 &= -3 \end{aligned}$

mult. by scalar $\frac{1}{3}$ in second equation:

$\Longleftrightarrow$ $\begin{aligned} x_1 - x_2 &= 4 \\ x_2 &= -1 \end{aligned}$)

now easy to see $x_1 = 3$.

(add $x_2 = -1$ back to $x_1 - x_2 = 4$)

Now just axiomatize this process.

Shorthand, act on matrix and on solution vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$:

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 3 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

Call such a matrix associated to a system $A\underline{x} = \underline{b}$, written $[A \mid \underline{b}]$, an augmented matrix. We have 3 basic operations on them (suggested by our previous example)

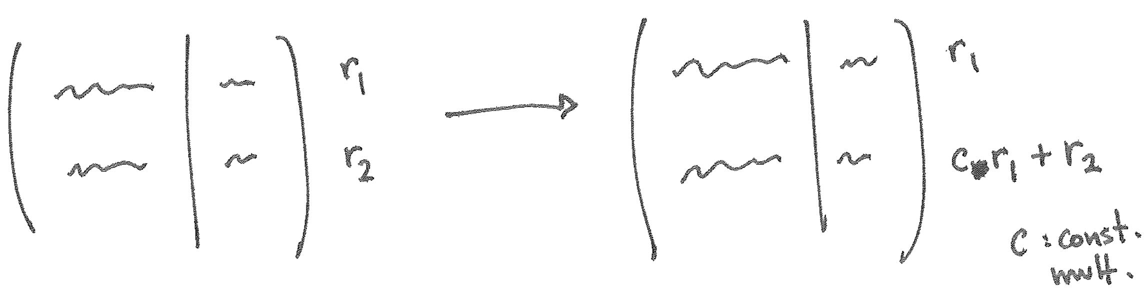
(1) multiply by a non-zero scalar in row.

(2) Add a multiple of one row to another row.

(3) Exchange rows. ← not necessary for solving equations, just puts augmented matrix in more prescribed form.

<u>Theorem</u>: These three operations preserve the set of solutions to the associated system of equations.

<u>pf</u>: Clear that (1) and (3) do this. For (2), if we take



if $\underline{x}$ is sol'n to first system, then sol'n for second.

And map is reversible. If we let $r_1' = r_1$, $r_2' = c r_1 + r_2$ then map $r_1' \mapsto r_1'$, $r_2' \mapsto r_2' - c r_1'$ to go back.

<u>Claim</u>: Any (augmented) matrix can be put into form:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

① every row has first non-zero entry equal to 1 (pivot)

② the pivots move down to the right as we go down rows

③ pivot is only non-zero entry in its column.

④ rows of zeros, if any, are at bottom

This is called "echelon form" or "row echelon form"

(Theorem 2.1.7 in H-H).

pf of claim: Give an algorithm: (1) Searching left to right, find first non-zero column. Pick a row with non-zero entry in this column, move it to the top, scale so the first entry is 1.

Running example:

$$\begin{bmatrix} 5 & 10 & 3 & 2 \\ 2 & 4 & 8 & 2 \\ 1 & 2 & 4 & 1 \end{bmatrix}$$

Pick third row, since we don't have to scale:

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 4 & 8 & 2 \\ 5 & 10 & 3 & 2 \end{bmatrix}$$

(2) Use multiples of new first row to make 0's below the new pivot 1.

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -17 & -3 \end{bmatrix}$$

(3) Find next column with non-zero entries below the first row. ← this column.

Pick a row with non-zero entry below first row in this column. Move it to the next (i.e. second) row.

Scale the first entry to be 1.

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 0 & 1 & 3/17 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(4) Just as in step (2), make all other entries in pivot column = 0.

$$\begin{bmatrix} \underline{1} & 2 & 0 & 5/17 \\ 0 & 0 & \underline{1} & 3/17 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Discuss relation to system of eqn's.

(5) Repeat until in echelon form.
(In our example, done)