On Friday, we claimed the following theorem:

A is invertible (i.e. there exists C with $AC = CA = \mathrm{Id}$.)
call C by A^{-1}

if and only if, for every $\underline{b} \in \mathbb{R}^m$, $A\underline{x} = \underline{b}$ has a

unique sol'n. (Already know this latter statement true iff
$\tilde{A} = I_n$ identity matrix)

(⇒) Suppose A invertible.

Pf: If $A\underline{x} = \underline{b}$, then $\underbrace{A^{-1}A}_{I_n}\underline{x} = A^{-1}\underline{b}$ (left inverse property)

so $\underline{x} = A^{-1}\underline{b}$. so showed that if $A\underline{x} = \underline{b}$ has sol'ns, then
unique sol'n is $\underline{x} = A^{-1}\underline{b}$.

Still have to prove solutions exist.
(in our case, check that $A^{-1}\underline{b}$ is always a solution)

$$A(A^{-1}\underline{b}) \underset{\substack{\text{assoc. of} \\ \text{matrix mult.}}}{=} (AA^{-1})\underline{b} = I_n\underline{b} = \underline{b}.$$ (right inverse property).

Next show (⇐).

What about converse: If $A\underline{x} = \underline{b}$ has unique sol'n for every $\underline{b}$

(i.e. A reduces to $\tilde{A} = I_n$) is it true that A is invertible?

YES! Pick $\underline{b} = \vec{e}_i$. Then $\exists$ $\underline{c}_i$ with $A \cdot \underline{c}_i = \vec{e}_i$

Make matrix from these $\underline{c}_i$'s as column vectors: $C = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}$ then $A \cdot C = Id$.

So A has a right inverse.

Does A have C as a left inverse?

$$\underbrace{[\, A \mid I_n \,]}_{\text{augmented matrix with } n \text{ vectors } \vec{e}_i \text{ simultaneously}} \xrightarrow[\text{reduce}]{\text{row}} \underbrace{\Big[\, I_n \;\Big|\; \underbrace{\begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}}_{= C} \,\Big]}_{\text{since each } A\underline{x} = \vec{e}_i \text{ has unique sol'n } \underline{c}_i.}$$

with elementary matrix product E such that

$E \cdot A = I_n$, $E I_n = C$, so $E = C$

and thus $C \cdot A = I_n$, a left inverse.

We can even use this to calculate inverses...

Do simultaneously: row reduction on A, keeping track of its effect on I_n.

Do example in 2×2 case

Given $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, ask

whether $\underline{b} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_i \in \mathbb{R}\}$

set of all linear combinations.

(a subspace)

$\updownarrow$

Linear equations: Does there exist column vector $\underline{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$ such that

$A \cdot \underline{c} = \underline{b}$ where A = matrix with columns $\vec{v}_i$ $= \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{pmatrix}$

(since $A \cdot \underline{c} = c_1 \cdot \vec{v}_1 + \dots + c_k \vec{v}_k$)

$\updownarrow$

Linear transformations: A : $n \times k$ matrix $\longleftrightarrow$ linear transformation T : $\mathbb{R}^k \to \mathbb{R}^n$

asking whether $\underline{b}$ is in the image of T.

To answer this question, use row reduction, solve system $A \cdot \underline{c} = \underline{b}$.

e.g. $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, Is $\underline{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

Ans: Examine augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$$ trouble...

No! System has no sol'n since has pivot in $\tilde{b}$.

What went wrong? Seems like 3 vectors in $\mathbb{R}^3$ should have $\text{Span}\{\vec{v}_i\}_{i=1}^3 = \mathbb{R}^3$. Here check $\vec{v}_1 = -\vec{v}_2 + 2\vec{v}_3$.

So anything in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is actually in Span$\{\vec{v}_2, \vec{v}_3\}$:

Indeed given $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \underset{\substack{\text{use relation} \\ \text{on } \vec{v}_1}}{=} c_1(-\vec{v}_2 + 2\vec{v}_3) + c_2\vec{v}_2 + c_3\vec{v}_3$

$$= \underbrace{(c_2 - c_1)}_{c_2'}\vec{v}_2 + \underbrace{(2c_1 + c_3)}_{c_3'}\vec{v}_3$$

$\in$ Span$\{\vec{v}_2, \vec{v}_3\}$.

Further question: Is it unique?
(if a linear combination exists)

Example: $\underline{d} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$. Then $\vec{v}_1 + \vec{v}_2 = \underline{d}$ so in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

But looking at $\tilde{A}$, then must be infinitely many solutions.

Another is $2 \cdot \vec{v}_3$. So $\vec{v}_1 + \vec{v}_2 = 2\vec{v}_3$. i.e. $\vec{v}_1 = 2\vec{v}_3 - \vec{v}_2$.

Plan: Remove $\vec{v}_1$. Now ask whether element in Span$\{\vec{v}_2; \vec{v}_3\}$ is unique.

$$A' = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \tilde{A}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so if a sol'n exists it is unique.

In fact, this shows in general that if A is matrix with columns $\vec{v}_1, \dots, \vec{v}_k$

then $\vec{v} \in$ Span$\{\vec{v}_1, \dots, \vec{v}_k\}$ has unique linear comb. $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{v}$

if and only if $\tilde{A}$ has pivots in all columns.