Local maxima / minima of diff. functions.

In one-variable $f: \mathcal{U} \subseteq \mathbb{R} \to \mathbb{R}$, we find critical points

(places where f not diff., e.g. sharp point — exclude those for now and just discuss f diff. on all of $\mathcal{U}$.

AND places where $f'(x) = 0$.)

Further test to decide if f has local max/min —

Second derivative test : if $f'(a) = 0$, $\underbrace{f''(a) > 0}_{\text{concave up}}$ $\rightsquigarrow$ local min at a

if $f'(a) = 0$, $\underbrace{f''(a) < 0}_{\text{concave down}}$ $\rightsquigarrow$ local max at a.

if $f'(a) = 0$, $f''(a) = 0$ $\rightsquigarrow$ inconclusive.

e.g. $f(x) = \begin{cases} x^4 & \to 0 \text{ is min} \\ -x^4 & \to 0 \text{ is max} \\ x^3 & \to 0 \text{ is neither} \end{cases}$

Same plan for $f: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$, diff. on $\mathcal{U}$

Say f has critical point at $\underline{a} \in \mathbb{R}^n$ if $[Df(\underline{a})] = [0]$ $\rightsquigarrow$ the $1 \times n$ 0-matrix.

i.e. tangent hyperplane to graph is parallel to $\mathbb{R}^n$ hyperplane.

picture : $f: \mathbb{R}^2 \to \mathbb{R}$ with $[Df(\underline{a})] = [0 \;\; 0]$

then tangent plane has equation $z = f(a)$, i.e. a horizontal plane ; parallel to xy-plane

Two remarks : ① Solve for which $\underline{a} \in \mathbb{R}^n$ have $[Df(\underline{a})] = [0]$ using Newton's method.

② What about second derivative test?

Rephrase second derivative test as asking about quadratic coeff. in Taylor expansion of $\underline{a}$.

What is analogue of quadratic term in multi-variable expansion?

All terms whose total degree is 2.

E.g. $f : \mathbb{R}^2 \to \mathbb{R}$, $P^2_{f,\underline{a}}$ has degree two terms $a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2$.

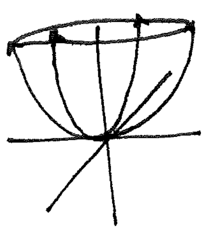
"quadratic form" ↑ not quadratic function

quadratic form means all monomials have degree exactly 2.

Easy examples: $x^2 + y^2 = Q(x,y)$

Graph this by looking at slices, like

$x = 0$, $y = 0$, $y = x$



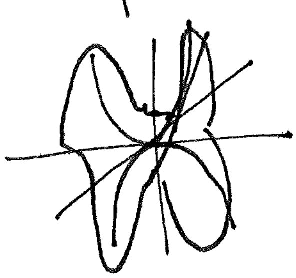
So if $P^2_{f,\underline{a}}$ had top terms of form x^2+y^2, suspect that f has local min at $\underline{a}$

Similarly $-x^2-y^2 = Q(x,y)$ flips graph over x-y plane.

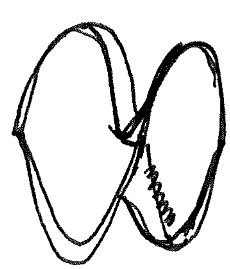
suggests f would have local max at $\underline{a}$ if these were top terms in $P^2_{f,\underline{a}}$.

What about $Q(x,y) = x^2 - y^2$?

Graph a little hard to picture. Again think of slices.



"saddle"



Move in x direction increase, in y direction decrease so this is neither max ~~nor~~ min.

What about arbitrary looking top form like

$x^2 + 2xy + y^2$? Write as $(x+y)^2$. (≥ 0, suggests if this appeared in $P^2_{f,\underline{a}}$, that $\underline{a}$ is local min.)

What about $x^2 + xy + y^2$?

Plan : complete the square.

$$x^2 + xy + \left(\frac{y}{2}\right)^2 = \left(x + \frac{y}{2}\right)^2$$

so
$$x^2 + xy + y^2 = x^2 + xy + \left(\frac{y}{2}\right)^2 + \frac{3y^2}{4}$$
$$= \left(x + \frac{y}{2}\right)^2 + \left(\frac{\sqrt{3}y}{2}\right)^2$$

whereas
$$x^2 + xy - y^2 = (x + y/2)^2 - \left(\frac{\sqrt{5}y}{2}\right)^2 \leftarrow$$
saddle in coords $x + y/2$, $\frac{\sqrt{5}y}{2}$.

Could we have done it differently? No.

Theorem : Given quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, $\exists$ $m = k + l$ linearly indep. linear functions $\alpha_1, \dots, \alpha_m : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$Q(\underline{x}) = \alpha_1(\underline{x})^2 + \dots \alpha_k(\underline{x})^2 - \alpha_{k+1}^2(\underline{x}) - \dots - \alpha_{k+l}(\underline{x})^2$$

And # plus signs $= k$ is independent of choice of α_i
minus signs $= l$ (intrinsic to Q).

Call the pair (k, l) the "signature" of Q. (Better : "footprint")

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linearly independent linear functions $\alpha_1, \dots, \alpha_m$:

if $\exists$ $c_1, \dots, c_m$ with $c_1\alpha_1 + \dots + c_m\alpha_m = 0$ identically as functions, for all values $\underline{x}$

then $c_1 = 0, \dots, c_m = 0$.

Check this by finding out if $n \times n$ matrix made from α_i's has full rank.

prove that such form is possible by induction.

1 variable quadratic form is $c \cdot x^2$, $c \in \mathbb{R}$. ✓.

For n variable form, two cases:

(a) $\exists$ term of form $c_i \cdot x_i^2$ for some $c_i \in \mathbb{R}$, $i \in [1, \dots, n]$.

then gather terms with x_i and complete square.

what's left is in $n-1$ vars, so use induction hypothesis.

Easy to check resulting linear functions are independent by evaluating at $\vec{e}_i$.

Example: $Q(\underline{x}) = x_1^2 + 2x_1x_2 + 6x_1x_3 - x_3^2$

Pick out terms with x_1: $x_1^2 + (2x_2 + 6x_3)\, x_1$ $\rightsquigarrow$ complete the square: add and subtract $\left(\frac{1}{2}\underbrace{(2x_2 + 6x_3)}_{x_2 + 3x_3}\right)^2$

$\|$

$(x_1 + x_2 + 3x_3)^2 - (x_2 + 3x_3)^2 - x_3^2$. ✓

(b) No square terms in Q. Just mixed quadratic terms $c_{i,j}\, x_i x_j$ with $i \neq j$.

Do initial substitution $x_i \longmapsto x_j + u$

if ~~for~~ some non-zero $c_{i,j}\, x_i x_j$ is in Q.

this monomial becomes $c_{i,j}\,(x_j^2 + u x_j)$. Revert to previous case, now with vars $x_1, \dots, x_n, u$

$\uparrow$

not x_i.