In other words, $\underline{b}$ is honest local extremum for composition $f \circ \tilde{g}$.

On other hand, we compute $D(f \circ \tilde{g})$ by chain rule:

$$= \left[Df(\underbrace{\tilde{g}(\underline{b})}_{\underline{c}}) \right] \left[D\tilde{g}(\underline{b}) \right] \quad \text{so if this} = 0,$$

$$\operatorname{Ker}\left[Df(\underline{c}) \right] \supseteq \operatorname{Im}\left[D\tilde{g}(b) \right] = \text{graph of } [Dg(b)]$$

$$= T_c M. \quad //$$

Problem yet to be solved: Find $\underline{c} \in \mathcal{U} \cap M$

$\mathcal{U}$: open set w/ f defined

M: smooth k-Manifold in $\mathbb{R}^n$

such that $\operatorname{Ker}\left[Df(\underline{c}) \right] \supseteq \operatorname{Ker}\left[DF(\underline{c}) \right]$.

$f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$

function we're optimizing

so $Df(\underline{c})$ is $1 \times n$

$F : \mathbb{R}^n \to \mathbb{R}^{n-k}$

so $DF(\underline{c})$ is

$n-k \times n$.

As linear transformations, when is $\operatorname{Ker}[\beta] \supseteq \operatorname{Ker}[A]$ $\quad \beta : 1 \times n$, $A : m \times n$. ?

Think of A is m $1 \times n$ matrices $\alpha_1, \dots, \alpha_m$

claim: $\operatorname{Ker}[\beta] \supseteq \operatorname{Ker}[A] \iff \exists\ \lambda_1, \dots, \lambda_m \in \mathbb{R}$ s.t.

$$\beta = \lambda_1 \alpha_1 + \dots + \lambda_m \alpha_m$$

($\Leftarrow$) is immediate

($\Rightarrow$) next time...

($\Rightarrow$) of Lemma 3.7.11. By contrapositive: Assume $\beta \notin$ ~~__~~ $\text{Span}(\alpha_1, \dots, \alpha_m)$

with α_i : rows of A, our $m \times n$ matrix.

Show $\exists$ $v \in \ker(A)$ but not in $\ker(\beta)$

(i.e. $\ker(A) \not\subseteq \ker(\beta)$).

First, take maximally linear indep. subset of α_i, call it $\alpha_1, \dots, \alpha_k$.

(i.e. $\alpha_{k+1}, \dots, \alpha_m$ in $\text{Span}(\alpha_1, \dots, \alpha_k)$) so $\ker(A) = \ker \underbrace{\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}}_{k \times n \text{ matrix.}}$

Since $\beta \notin \text{Span}(\alpha_1, \dots, \alpha_m) = \text{Span}(\alpha_1, \dots, \alpha_k)$

then set $\text{Span}(\alpha_1, \dots, \alpha_k, \beta)$ is $k+1$ dimensional.

i.e. $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ \beta \end{bmatrix} : \mathbb{R}^n \longrightarrow \mathbb{R}^{k+1}$ is onto. (Image is column span but dim col. span = dim row span)

Pick $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^{k+1}$, with sol'n $\vec{v}$ s.t. $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ \beta \end{bmatrix} \underline{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. This is desired $\underline{v}$. //

Corollary (Lagrange multipliers) $\underline{a}$ is constrained critical point of f on manifold $M \longleftrightarrow F(\underline{z}) = 0$, $F: \mathbb{R}^n \to \mathbb{R}^m$ if and only if $\exists\ \lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that $[Df(\underline{a})] = \lambda_1 [DF_1(\underline{a})] + \dots + \lambda_m [DF_m(\underline{a})]$

(as this gives containment of kernels)

Back to circle example and $f(x, y) = xy$.

$Df(x_0, y_0) = [y_0, x_0]$

$\underset{F_1}{\overset{\parallel}{F}} : \mathbb{R}^2 \to \mathbb{R}^1 : x^2 + y^2 - 1 = 0.$

$DF_1(x_0, y_0) = [2x_0, 2y_0].$

so need λ_1 s.t. $[y_0, x_0] = \lambda_1 [2x_0, 2y_0]$

$$\left.\begin{aligned} y_0 &= 2\lambda_1 x_0 \\ x_0 &= 2\lambda_1 y_0 \end{aligned}\right\} \quad \lambda_1 = \frac{x_0}{2y_0} = \frac{y_0}{2x_0} \Rightarrow x_0^2 = y_0^2$$

$x_0^2 + y_0^2 = 1$

so $x_0 = \pm y_0$.

Higher dimensional example: $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}$

$(x, y, z) \longmapsto xyz$.

with $M \longleftrightarrow F : x + 2y + 3z - 1 = 0$ (plane in $\mathbb{R}^3$)

Need (x_0, y_0, z_0) s.t. $[y_0 z_0, x_0 z_0, x_0 y_0] = \lambda \cdot [1, 2, 3]$

and $x_0 + 2y_0 + 3z_0 - 1 = 0$.

Writing it out:

$$\begin{aligned} y_0 z_0 &= \lambda \\ x_0 z_0 &= 2\lambda \\ x_0 y_0 &= 3\lambda \\ x_0 + 2y_0 + 3z_0 - 1 &= 0 \end{aligned}$$

$2y_0 z_0 = x_0 z_0$

$\Downarrow$

$z_0 = 0$ OR $2y_0 = x_0$

$3y_0 z_0 = x_0 y_0$

$\Downarrow$

$y_0 = 0$ or $3z_0 = x_0$

combine into here: if $y_0, z_0 \neq 0$:

$x_0 + x_0 + x_0 - 1 = 0$

$x_0 = 1/3$

$y_0 = 1/6$

$z_0 = 1/9$

$F(x_0, y_0, z_0) = \frac{1}{162}$

max

---

if z_0 or $y_0 = 0$ then $\lambda = 0$

which implies another of remaining vars $= 0$.

get add'l critical points (using plane equation: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix}$.

Saddles.

All give $F = 0$.