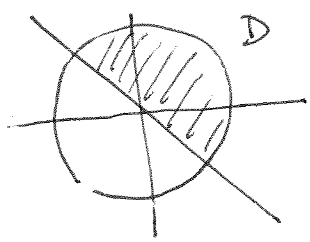
Examples: $z^2 = x^2 + y^2$ cut by plane $z = 1 + x + y$.

Call result C. Find closest point to origin on C.

Let D be domain bounded between $x + y = 0$ and $x^2 + y^2 = 1$



Maximize/minimize $f\begin{pmatrix} x \\ y \end{pmatrix} = xy$ on D

OR

$f\begin{pmatrix} x \\ y \end{pmatrix} = x + 5xy$

Critical points of $f(x, y, z) = xyz$ on surface $x + y + z^2 = 16$.

Is there a maximum?

Find max of $x_1 \cdots x_n$ subject to constraint $x_1^2 + 2x_2^2 + \cdots + nx_n^2 = 1$.

$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + z^2$ sphere $\overset{F}{x^2 + y^2 + (z-1)^2 = 1}$

Find shortest distance between ellipse $x^2 + 2y^2 = 2$

and line $x + y = 2$.

f : square of distance.

Constraint : $\mathbb{R}^4 \to \mathbb{R}^2$ on both curves.

One theoretical application of Lagrange multipliers is the spectral theorem —

Remember that an eigenvector $\vec{v}$ for transformation A is vector for which $\exists \lambda$ s.t. $A\vec{v} = \lambda \vec{v}$ (i.e. A acts by scaling)

($A : \mathbb{R}^n \to \mathbb{R}^n$ linear)

If we can find $v_1, \dots, v_n$ a basis of $\mathbb{R}^n$ with v_i : eigenvectors.

Q : change of basis matrix from $\vec{e}_i$ to $\vec{v}_i$ then

$Q^{-1} A Q$ is diagonal matrix $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ λ_i : eigenvalues.

Big question: when can this be done?

Not always. e.g. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is rotation by $90°$. No non-zero vector is scaled under such a rotation (or any other rotation $\neq 180°$.)

Spectral Theorem : If A is a symmetric, real, $n \times n$ matrix, $(A = A^T)$ then $\exists$ basis of eigenvectors $v_1, \dots, v_n$ (in fact, chosen to be orthonormal)

size 1, ~~mutually~~ pair-wise orthogonal.

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pf. uses Lagrange multipliers. Rough idea :

A : symmetric $\longleftrightarrow$ Quadratic form Q_A where

$$Q_A(\underline{z}) = (z_1 \;\cdots\; z_n) \cdot A \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

e.g. $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ $\longrightarrow$ $Q_A(z_1, z_2) = (z_1 \; z_2) \begin{pmatrix} a & b \\ b & d \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

2x2

bijection because Q_A has 3 coefficients, A has 3 distinct entries.

$$(z_1 a + z_2 b \quad z_1 b + z_2 d) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= z_1^2 a + z_1 z_2 b + z_2 z_1 b + z_2^2 d$$

Infer properties of A from those of Q_A, viewed as map $\mathbb{R}^n \to \mathbb{R}$,

where we impose additional constraints.

Just do beginning: Q_A restricted to unit $\underbrace{\text{sphere}}_{\text{compact set,}}^{S}$ $\underbrace{|\underline{x}|^2}_{F\,:\,|\underline{x}|^2 - 1 = 0} = 1$.

so $Q_A|_S$ has max / min.

compute $\left[DQ_A(\underline{c})\right]$, $\left[DF(\underline{c})\right]$

more elegant to write them as linear transformations:

$$\left[DF(\underline{c})\right] \cdot \underline{h} = 2\,\underline{c} \cdot \underline{h} \quad \text{or} \quad 2\,\underline{c}^T \underline{h}.$$

$$\left[DQ_A(\underline{c})\right] \cdot \underline{h} = \underline{c} \cdot A\underline{h} + \underline{h} \cdot A\underline{c}$$

$$= 2\,\underline{c}^T A\,\underline{h}$$

maximum $\underline{c}$ has λ_1 such that $2\,\underline{c}^T \cdot A = \lambda_1\, 2\,\underline{c}^T$

$\Rightarrow \underbrace{A^T}_{\substack{\shortparallel \\ A \text{ since} \\ A \text{ assumed} \\ \text{symmetric.}}} \cdot \underline{c} = \lambda_1 \underline{c}$ so $\underline{c}$ is eigenvector with length 1 since it is on manifold.

Call it v_1.

Repeat.

Next constraint: Lie on

S : sphere + have $\underline{x} \cdot \underbrace{v_1 = 0}_{\text{orthogonal.}}$