Open sets in $\mathbb{R}^n$. (geometric intuition better if we think of points rather than vectors)

First, open ball $B_r(\underline{x})$ = set of points less than r distance from x

$= \{ y \in \mathbb{R}^n \mid |\underline{x} - \underline{y}| < r \}$

Then open set of $\mathbb{R}^n$: $\mathcal{U}$ is open in $\mathbb{R}^n$ if, for every $\underline{x} \in \mathcal{U}$

$\exists$ some r such that $B_r(\underline{x}) \subseteq \mathcal{U}$.

example of "neighborhood of x" (any set containing $B_r(\underline{x})$ for some r)

Discuss examples:

① open interval (a,b) is open in $\mathbb{R}$. Warning: not open in $\mathbb{R}^2$ since ball is taken in given ambient space.

② Any open ball is open set.

Open "shapes" like square w/o boundary. Do some non-examples.

We want functions to be defined on open sets, so that nbhds exist for any point, and we may approach in any direction.

closed set: complement of open set. So $C \subseteq \mathbb{R}^n$ closed if $\mathbb{R}^n - C$ is open.

We can also define closure / interior of set $\rightsquigarrow$

idea: Given $\mathcal{U} \subseteq \mathbb{R}^n$, find smallest closed subset containing $\mathcal{U}$

largest open subset contained in $\mathcal{U}$.

make definitions (and in HW, check that the aforementioned characterizations are true)

<u>closure</u>: Given $\mathcal{U} \subseteq \mathbb{R}^n$, closure $\overline{\mathcal{U}} :\overset{\text{def}}{=} \{ \underline{x} \in \mathbb{R}^n \mid B_r(\underline{x}) \cap \mathcal{U} \neq \phi \; \forall\, r>0 \}$

<u>Interior</u>: Given $\mathcal{U} \subseteq \mathbb{R}^n$, $\mathring{\mathcal{U}}$: interior $\overset{\text{def}}{=}$

$$\{ \underline{x} \in \mathbb{R}^n \mid \exists\, r>0 \text{ with } B_r(\underline{x}) \subseteq \mathcal{U} \}$$

in either case (again HW) you can characterize the points added/subtracted:

<u>boundary</u> Given $\mathcal{U} \subseteq \mathbb{R}^n$, $\partial\mathcal{U}$: boundary of $\mathcal{U}$ =

$$\{ \underline{x} \in \mathbb{R}^n \mid \text{every nbhd. of } \underline{x} \text{ has non-empty intersection with } A,\ \underbrace{\mathbb{R}^n - A}_{\text{complement of } A} \}$$

Then

$$\overline{\mathcal{U}} = \mathcal{U} \cup \partial\mathcal{U}, \quad \mathring{\mathcal{U}} = \mathcal{U} - \partial\mathcal{U}.$$

so $\partial\mathcal{U} = \overline{\mathcal{U}} - \mathring{\mathcal{U}}$.

Do more examples... open unit disk − origin, $|y| < x^2$: between two parabolas

<u>limits</u>: limit of sequence (book rightly points out that need to get order of quantifiers straight)

$\{\underline{a}_i\}_{i=1}^{\infty}$, $\underline{a}_i \in \mathbb{R}^n$

converges to a point $\underline{a}$ if, for every $\epsilon>0$, $\exists\, M$

such that, if $m \geq M$, $|\underline{a}_m - \underline{a}| < \epsilon$.

Often described as a game - challenged with an ϵ, need to produce M.

Simple examples:

In $\mathbb{R}^1$, $\left\{ \frac{1}{i} \right\}_{i=1}^{\infty} = \{ 1, 1/2, 1/3, \dots \}$ Intuitively clear that limit is 0. But prove this.

If challenged with $\varepsilon = .12$, then

choose $M = 9$ since then $a_M = 1/9$ and $| 1/9 - 0 | = 1/9 = .111\dots$

For $\varepsilon = .12$, any $M \geq 9$ would do (or even $M = 8$ if we use $\underset{\substack{\uparrow \\ \text{strict}}}{m > M}$)

But we need to prove for all ϵ, so we need formula for choosing M for any ϵ. ~~[illegible]~~ In our case, $\frac{1}{i}$ is strictly decreasing, so to get $\left| \frac{1}{i} - 0 \right| < \epsilon \quad \forall\, i \geq M$, ~~[illegible]~~ just need $\left| \underbrace{\frac{1}{M} - 0}_{= \frac{1}{M}} \right| < \epsilon$

i.e. pick any $M > \frac{1}{\varepsilon}$. ✓

Example 2: In $\mathbb{R}^2$, $\left\{ \begin{pmatrix} \frac{1}{i} \\ \frac{i}{i+1} \end{pmatrix} \right\}_{i=1}^{\infty}$. Again, we're confident that limit is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Now $\left| \begin{pmatrix} \frac{1}{i} \\ \frac{i}{i+1} \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|$ more complicated. It is equal to

$$\sqrt{ \left(\frac{1}{i} - 0 \right)^2 + \left(\frac{i}{i+1} - 1 \right)^2 }.$$

We could expand this, try to argue that $i \geq M$, then

$$\left| \underline{a}_i - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \leq \left| \underline{a}_M - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|$$

then find formula for M in terms of ϵ using expression.

Better: Prove a result that limit of points/vectors in $\mathbb{R}^n$ converges if and only if it converges in each component. So reduce to question in $\mathbb{R}^1$.

Now in proof, we need to use that there is M_i s.t. $m > M_i := M_i(\varepsilon_i)$

$$|\underbrace{(\underline{a}_m)_i}_{\substack{i\text{th component}\\ \text{of } \underline{a}_m \in \mathbb{R}^n}} - \underline{a}_i| < \varepsilon_i \quad \text{for each } i = 1, \dots, n.$$

To show that ~~something~~ $|\underline{a}_m - \underline{a}|$ can be made arbitrarily small for some M.
(i.e. $< \varepsilon$, any ε)

$$|\underline{a}_m - a| = \sqrt{((\underline{a}_m)_1 - a_1)^2 + \cdots + ((\underline{a}_m)_n - a_n)^2} \qquad (*)$$

If we choose $M = \underset{1 \le i \le n}{\mathrm{Max}} \; M_i$, then we can guarantee each component $< \varepsilon_i$

so $(*) \le \underbrace{\sqrt{\varepsilon_1^2 + \cdots + \varepsilon_n^2}}_{\left|\begin{pmatrix}\varepsilon_1\\ \vdots \\ \varepsilon_n\end{pmatrix}\right|}$.

If we are given $\varepsilon > 0$, want to find M with $(*) < \varepsilon$,

then pick ε_i's so that

$$\left|\begin{pmatrix}\varepsilon_1\\ \vdots \\ \varepsilon_n\end{pmatrix}\right| \le \varepsilon$$

and then desired $M = \underset{i}{\mathrm{Max}} \; M_i(\varepsilon_i)$.

One cute way to do this:

$\varepsilon_i = \dfrac{\varepsilon}{\sqrt{n}}$, then $\left|\begin{pmatrix}\varepsilon_1\\ \vdots \\ \varepsilon_n\end{pmatrix}\right| = \varepsilon$.

(style points for elegant choice ...)

Alternately, choose $\varepsilon_i = \varepsilon$ for $i = 1, \dots, n$, get

$$|\underline{a}_m - a| < \sqrt{n \cdot \varepsilon^2} = \sqrt{n} \cdot \varepsilon \quad \text{if} \quad m > M = \underset{i}{\mathrm{Max}} \; M_i(\varepsilon)$$

clear that since $\sqrt{n}\,\varepsilon$ can made arbitrarily small.

Proof of opposite direction in "iff" statement of Prop 1.5.13 is much easier. Leave you to read it.