On Wednesday, final ingredient before taking up derivatives is MVT.

proved it except for statement:

$f : [a,b] \to \mathbb{R}$ continuous, then f attains max/min on this closed interval.

More general statement can be made, but need definitions.

<u>Bounded</u>: We say $X \subset \mathbb{R}^n$ is bounded if it is contained in a ball of finite radius. (H-H take ball centered at origin.) Why not?

<u>Compact</u>: We say $C \subset \mathbb{R}^n$ is compact if it is closed + bounded.

Note for example (a,b) is bounded in $\mathbb{R}$, but not closed.

$[a,\infty)$ is closed in $\mathbb{R}$, but not bounded.

while $[a,b]$ is both i.e. compact.

Then more general statement (<u>Theorem 1.6.9</u>): $C \subseteq \mathbb{R}^n$ compact

$f : C \to \mathbb{R}$ continuous, then $\exists\, \underline{M} \in C$ s.t. $f(\underline{M}) \geq f(\underline{x})$ $\forall\, \underline{x} \in C$

and $\underline{m} \in C$ s.t. $f(\underline{m}) \leq f(\underline{x})$ $\forall\, \underline{x} \in C$.

To prove this, first prove Bolzano-Weierstrass Thm.

Now we can state:

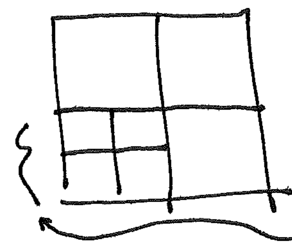
<u>Theorem</u> (Bolzano-Weierstrass) Every sequence $\{\underline{x}_j\}$ on compact set C has convergent subsequence $\{\underline{x}_{i_1}, \underline{x}_{i_2}, \dots\}$ whose limit lies in C.

<u>pf</u>: C is assumed closed, so if $\{\underline{x}_{i_k}\}_k$ converges, then must converge to a point in C. So we just have to show a convergent subsequence exists.

Rough <u>idea</u>: C is bounded, so finite volume. Infinitely many points in finite volume must accrue somewhere.

So put C inside big n-dim'l box in $\mathbb{R}^n$. Form smaller boxes by cutting in half along all dimensions. In $\mathbb{R}^2$:

NOTE: technical issue about whether to include bdry of each square



Suppose this was B_1

one of these 2^n boxes must contain infinitely many points.

We don't know which, but we just need to show conv. subsequence exists.

Call this B_1. Pick $\underline{x}_{i_1}$ in B_1 from $\{\underline{x}_j\}$. Now chop all coordinates defining B_1 in half. Repeat argument, choosing B_2 with ∞-ly many points and $\underline{x}_{i_2}$ in B_2.

We claim this sequence $\{\underline{x}_{i_1}, \underline{x}_{i_2}, \dots\}$ converges.

Indeed, given any $\epsilon > 0$, we just find a box B_M that fits in a ball of radius $\epsilon/2$. then this is desired M so that $|\underline{x}_m - \underline{a}| < \epsilon$, if $m \geq M$. (What is $\underline{a}$ by the way? We don't know. We just know it lives in all of the boxes.)

Book proof is even more concrete by choosing boxes that shrink by 10^{th}s so you can think of each box as giving decimal expansion of eventual limit.

Book has nice example illustrating issues with Bolzano-Weierstrass

Some sequences are easy to analyze. e.g. $\{\frac{1}{n}\}_{n=1}^{\infty}$ or even

$\{1, 1, 1/2, 1, 1/3, 1, 1/4, 1, \ldots\}$ but $\sin(x_n)$ for any sequence of real #'s x_n is an ∞ sequence in $[-1,1]$ so has conv. subsequence by Bolzano-Weierstrass thm.

However, if we try to find which cuts contain ∞-ly many pts, find it is very hard to do.

e.g. $x_n = 2 \cdot 10^n$

then $\sin(x_n)$ pos/neg. depending on n^{th} digit of π. Hard question...

On to proof of Thm. 1.6.9.

recall that a number S is the supremum of $f: \mathcal{U} \to \mathbb{R}$

if it is the <u>least upper bound</u> of the values $\{f(\underline{x}) \mid \underline{x} \in \mathcal{U}\}$

Write $S = \sup\limits_{\underline{x} \in \mathcal{U}} f$.

upper bound: a number $\underline{a}$ s.t.

$\underline{a} \geq f(\underline{x}) \quad \forall \ \underline{x} \in \mathcal{U}$.

least upper bound: if $\underline{b}$ is any other upper bound, $\underline{a} \leq \underline{b}$.

there is a corresponding notion of greatest lower bound.

<u>Property of real numbers</u>: Every nonempty subset $X \subset \mathbb{R}$ with upper bound has a least upper bound. (Thm 0.5.3) in book

Another statement for lower bounds.