 Name: KEY

3. Consider the set of solutions to the system of equations

$$x^3 + 2xy - z = 0$$
$$y^2 - xz + x = 0$$

a) Does this system of equations define a smooth 1-manifold in $\mathbb{R}^3$?

Our criterion for a manifold is that the Jacobian is onto. (If F is C^1 function which it is!) then implicit function theorem guarantees required implicit function exists in nbhd. of any point.

$$DF\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 3x^2+2y & 2x & -1 \\ -z+1 & 2y & -x \end{bmatrix}.$$

This is onto if column space has dimension 2. ("rank") (continued...)

b) Find the equation of the tangent ~~plane~~ line to the locus at $(x, y, z) = (1, 0, 1)$.

The simplest way to write the equation of the tangent line to this 1-manifold at $\underline{c} = (1,0,1)$ is:

$$\left[DF(\underline{c})\right] \cdot \begin{bmatrix} x-1 \\ y-0 \\ z-1 \end{bmatrix} = \underline{0}$$

(this will give tangent line as intersection of two planes in $\mathbb{R}^3$)

Here $\left[DF(\underline{c})\right] = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

c) At $(1, 0, 1)$, what are the possible independent variables defining an implicit function ϕ locally parametrizing the zero locus? Pick one and give the second order Taylor expansion for the resulting function ϕ at $(1, 0, 1)$.

Remember that we may reorder columns, so either x or y could be non-pivot variables for $\left[DF(\underline{c})\right]$ with $\underline{c} = (1,0,1)$. Pick y.

$$\phi : y \longmapsto \begin{bmatrix} x := \phi_1(y) \\ z := \phi_2(y) \end{bmatrix} \quad \text{with} \quad \phi'(0) = -\begin{bmatrix} 3 & -1 \\ 0 & -1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 0 \end{bmatrix}$$

Alternate question: Similar for $F : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ given by

$$F(x_1, x_2, x_3, x_4, x_5) = (2x_1+x_2+x_3+x_4-1,\ x_1x_2^3+x_1x_3+x_2^2x_4^2-x_4x_5,\ x_2x_3x_5+x_1x_3^2+x_4x_5^2)$$

(continued ...)

3(a) continued.

$$\begin{bmatrix} 2x & -1 \\ 2y & -x \end{bmatrix}$$ are dependent if det $= 0$:

$$-2x^2 + 2y = 0$$

$$\iff \quad y = x^2.$$

So $DF\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can fails to be onto only if

$y = x^2$. Check two other columns with $y = x^2$:

1st + 3rd with $y = x^2$:

$$\det \begin{bmatrix} 5x^2 & -1 \\ -z+1 & -x \end{bmatrix} = -5x^3 - z + 1 \overset{?}{=} 0$$

$$\Updownarrow$$

$$z = 1 - 5x^3.$$

1st + 2nd if $y = x^2$, $z = 1 - 5x^3$ does give 0.

Are these points on Manifold?

$y^2 - xz + x = 0$ becomes $x^2 - x(1 - 5x^3) + x = 0$

$$x^2 + 5x^4 = 0$$

$$\Rightarrow \quad x = 0. \quad \Rightarrow \quad y = 0, \; z = 1$$

But $(0,0,1)$ doesn't satisfy $x^3 + 2xy - z = 0$.

So it is smooth manifold!

3(c) continued...

So far, we've found the Taylor polynomial for $\phi : \mathbb{R} \to \mathbb{R}^2$

starts:

$$P^2_{\phi,0}(y) = \phi(0) + \phi'(0)(y-0) + \phi''(0)\frac{(y-0)^2}{2!}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 0 \end{bmatrix}(y-0) + \frac{\phi''(0)}{2}(y-0)^2$$

To solve for $\phi''(0)$, use the fact that

$$F\begin{pmatrix} \phi_1(y) \\ y \\ \phi_2(y) \end{pmatrix} = 0 \quad \text{in a nbhd. of } y=0, \text{ so}$$

$$F\begin{pmatrix} P^2_{\phi,0}(y)_1 \\ y \\ P^2_{\phi,0}(y)_2 \end{pmatrix} = 0 \quad (\text{up to order } 2)$$

[In fact, we could substitute P^2_F for F, but since F is already rather simple it isn't worth bothering...

ugly notation for two component functions of $P^2_{\phi,0}(y)$.

Here $P^2_{\phi,0}(y)_1 =$

$$1 - 2/3\, y + \frac{\phi_1''(0)}{2} y^2$$

and $P^2_{\phi,0}(y)_2 = 1 + \frac{\phi_2''(0)}{2} y^2$

Substituting into F:

$$\left(1 - 2/3\, y + \frac{\phi_1''(0)}{2}y^2\right)^3 + 2\left(1 - 2/3\, y + \frac{\phi_1''(0)}{2}y^2\right)y - \left(1 - \frac{\phi_2''(0)}{2}y^2\right) = 0 \pmod{y^3}$$

AND

$$y^2 - \left(1 - 2/3\, y + \frac{\phi_1''(0)}{2}y^2\right)\left(1 + \frac{\phi_2''(0)}{2}y^2\right) + \left(1 - 2/3\, y + \frac{\phi_1''(0)}{2}y^2\right) = 0 \pmod{y^3}$$

This looks much worse than it is. The equalities imply all coeffs of y, that is y^0, y^1, y^2, must vanish.

Since $\phi_i''(y)$'s appear only in y^2 terms, we need these alone:

Using our Pascal's triangle skills:

$$3\cdot\left(-\frac{2}{3}\right)^2 + \frac{\phi_1''(0)}{2} - \frac{4}{3} + \frac{\phi_2''(0)}{2} = 0$$

these cancel

AND

$$1 - \frac{\phi_1''(0)}{2} - \cancel{\frac{\phi_2''(0)}{2}} + \cancel{\frac{\phi_2''(0)}{2}} = 0$$

$$\Rightarrow \quad \phi_1''(0) = 2, \quad \phi_2''(0) = -2$$

(upon solving this very easy linear system.)

thus $P^2_{\phi,0}(y) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 0 \end{bmatrix} y + \begin{bmatrix} 2 \\ -2 \end{bmatrix} y^2.$