Definition of paving $\mathcal{P}$: Subset $X \subseteq \mathbb{R}^n$ is <u>paved</u> by $\mathcal{P} = \{P_i\}$

P_i : bounded subsets $\subseteq X$ s.t.

(1) $\bigcup_i P_i = X$

(2) $\mathrm{vol}_n(P_i \cap P_j) = 0$ if $P_i \neq P_j$.

(3) Any bounded subset of X intersects (only) finitely many P_i.

(4) $\forall P_i \in \mathcal{P}$, $\mathrm{vol}_n(\partial P_i) = 0$.

Then we want notion of shrinking set of ~~pavings~~ pavings. "Nested partition"

$\mathcal{P}_N$: sequence of pavings of $X \subseteq \mathbb{R}^n$ is "nested partition"

if ① every subset in $\mathcal{P}_{N+1}$ is contained in a subset of $\mathcal{P}_N$.

② the subsets of $\mathcal{P}_N$ shrink as $N \to \infty$: Precisely, to pts.

$$\lim_{N\to\infty} \sup_{P_i \in \mathcal{P}_N} \mathrm{diam}(P_i) = 0$$

$$\left(\mathrm{diam}(P_i) = \sup_{x,y \in P_i} |x-y| \text{ is the "diameter of } P_i\text{"} \right)$$

<u>Thm</u> (Independence of pavings by nested partitions)

$X \subseteq \mathbb{R}^n$. $\{\mathcal{P}_N\}$: nested partition of X.

If $f: \mathbb{R}^n \to \mathbb{R}$ integrable (w.r.t. dyadic pavings) then

$$\lim_{N\to\infty} U_{\mathcal{P}_N}(f \cdot 1_X) = \lim_{N\to\infty} L_{\mathcal{P}_N}(f \cdot 1_X) = \int_X f(\underline{x})\, |d^n \underline{x}|$$

Conversely, if upper and lower sums for $\{\mathcal{P}_N\}$ agree, then f integrable.

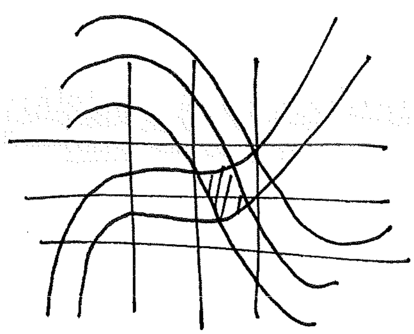
pf of independence of pavings:

First check 1_X is pavable by $\{\mathcal{P}_N\}$. True because of weird way we ~~[illegible]~~ defined nested partition of X. So that ∂X is contained in ∂P_i with $P_i \in \mathcal{P}_N$ for any N.

By definition, this is a set of measure 0. (So having nested partition of X means ∂X is somewhat nice)

Now main issue: how to compare dyadic pavings with arbitrary pavings? Want to relate their upper and lower sums.

In $\mathbb{R}^2$:



Issue: If paving $\mathcal{P}_N$ shares many dyadic cubes, hard to compare to Riemann sum over $\mathcal{D}_N$.

Idea: choose $N'' \gg N$ so that "most" of $P_i \in \mathcal{P}_{N''}$ are entirely inside cubes $\mathcal{D}_N$.

Write $U_{\mathcal{P}_{N''}}(f)$

$$= \sum_{P_i \cap \partial D_N = \phi} \sup_{P_i}(f) \cdot \mathrm{vol}_n P_i + \sum_{P_i \cap \partial D_N \neq \phi} \sup_{P_i}(f)\, \mathrm{vol}_n P_i$$

this is bigger that $-\sup_{\mathbb{R}^n}|f|$

need to make $\sum_{P_i \cap \partial D_N \neq \phi} \mathrm{vol}_n P_i$ small.

(choose N' so that $D_{N'} \cap \partial D_N$ small)

Determinants: Give a recursive definition or axiometric definition.

(determinants give volume expansion by action of linear map)

Recursive: Illustrate first column expansion recursive definition.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \text{then } \det(A) = 1 \cdot \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 4 \cdot \det\begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 7 \det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$$

take det. of submatrix of rows, columns away from $a_{i,1}$

signs alternate

So picture $1 \cdot \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$ as $\begin{pmatrix} 1 & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{pmatrix}$

You can take as definition the expansion along any fixed column. Not yet clear that they all produce same #. (be careful with signs)

Book uses symbol Δ_n for det. of $n \times n$ matrix.

$$\Delta_1([a]) = a. \qquad \Delta_n(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \cdot \Delta_{n-1}(A_{i,1})$$

$A_{i,1}$: $n-1 \times n-1$ matrix made from deleting row i and col. 1.

Thm: Δ_n is the unique function $f: \underbrace{(\mathbb{R}^n)^n}_{\text{n-tuples of vectors in } \mathbb{R}^n} \to \mathbb{R}$

(i.e. $n \times n$ matrix)

s.t.

1. f is linear in all components
2. f is antisymmetric
3. $f(I_n) = 1$. (Better: $f(e_1, \ldots, e_n) = 1$.)