General theorem on change of vars: $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ nice.

$$X \longmapsto Y$$

$$\int_Y f(\underline{y})\, |d^n \underline{y}| = \int_X (f \circ \Phi)(\underline{x})\, \left|\det\left(D\Phi_{(\underline{x})}\right)\right|\, |d^n \underline{x}|$$

nice: want $D\Phi$ invertible for all $\underline{x} \in X$, else mapping cubes in $\mathbb{R}^n$ to objects with vol. 0.

Also want Φ to take nested partition to nested partition.

Before describing "nice" precisely. do some examples.

Ex: $\displaystyle\int_A \frac{y}{x}\, |d(x,y)|$ where A is bounded by curves

$x^2 - y^2 = 1$

$x^2 - y^2 = 4$

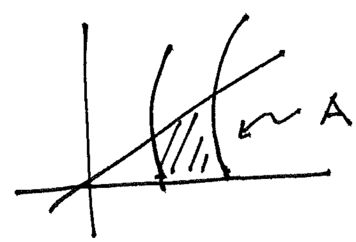
$y = 0$

$y = x/2$

Try: $u = x^2 - y^2$

$v = y/x$

We need the area element for x, y as a function of u, v.



$$|d(x,y)| = \left|\det\left(D_{x(u,v),\, y(u,v)}\right)\right|\, |d(u,v)|$$

Now what? Solve for x, y? No.

Just use $\det\left(D(x(u,v), y(u,v))\right) \cdot \det\left(\underbrace{D(u(x,y), v(x,y))}\right) = 1.$

by chain rule.

$$\begin{vmatrix} 2x & -2y \\ -y/x^2 & 1/x \end{vmatrix} = 2 - 2(y/x)^2 = 2 - 2v^2$$

put in limits of integration: $\displaystyle\int_0^{1/2}\int_1^4 \frac{v}{2(1-v^2)}\,du\,dv$

$$= -\frac{3}{4}(1-v^2)\Big|_0^{1/2}$$

$$= -3/4 \ln 3/4$$

(note bounds on v ensured that Jacobian $\dfrac{1}{2(1-v^2)}$ was never undefined)

Example: Express the iterated integral $\displaystyle\int_0^1\int_0^x dy\,dx$ as an iterated integral in u,v after changing $u=x+y$, $v=x-y$.

Want Jacobian $x(u,v)$, $y(u,v)$. Solve for x,y, or just invert:

$$\left| Du(x,y), v(x,y) \right| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2, \text{ so Jacobian for } \begin{matrix} x(u,v) \\ y(u,v) \end{matrix} = -1/2.$$

(then take abs. value)

What about bounds? Need to know image of boundary curves: $y=0$, $\underset{\sim}{y=x}$, $x=1$.

easy in this example: $v=0$.

Surest way: eliminate x,y from:

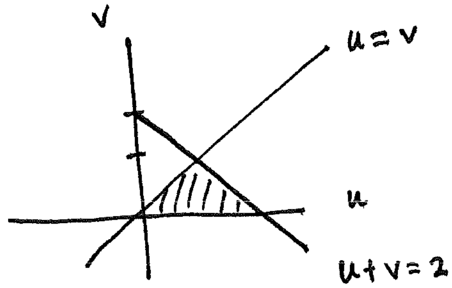
$$\left.\begin{aligned} u &= x+y \\ v &= x-y \\ y &= 0 \end{aligned}\right. \Rightarrow u=v.$$

Similarly

$$\left.\begin{aligned} u &= x+y \\ v &= x-y \\ x &= 1 \end{aligned}\right\} \begin{aligned} u &= 1+y \\ v &= 1-y \end{aligned} \Rightarrow u+v=2.$$

(or since this is linear, solve for x,y, substitute into boundary curves)

So



$$= \int_0^1 \int_v^{2-v} \frac{1}{2} \, du \, dv.$$

Full statement of change of vars. $\Phi : X \mapsto Y$.

Want X : compact with ∂X having volume 0.

Φ : $\underbrace{C^1}_{\text{cont-diff.}}$ function on open set $\mathcal{U} \supseteq X$ with Lipschitz derivative. (*)

Also : Want Φ injective on $X - \partial X$ (interior) and such that

$|D\Phi(\underline{x})|$ invertible at every $\underline{x} \in X - \partial X$.

Lingo from book : "Φ parametrizes Y by X".

(*) : $\exists M$: $\big| \underbrace{D\Phi(\underline{x}) - D\Phi(\underline{y})}_{\text{matrix distance for derivatives}} \big| \le M \, |\underline{x} - \underline{y}|$ $\forall$ $\underline{x}, \underline{y}$ on some subset.

(weaker than saying Φ is $\underbrace{C^2}_{\text{cont. twice diff.}}$ function.)

pf : 3 parts - ① show Φ takes pavings to pavings / nested partitions to nested partitions

- ② show that $\dfrac{\mathrm{vol}_n(\Phi(C))}{\mathrm{vol}_n(C)} \approx |\det(D\Phi(\underline{x}_C))|$ as $N \to \infty$.

(i.e. gets within $\pm\epsilon$ of ratio for any $\epsilon > 0$)

③ Mimic proof in linear case

For ①, need that $\Phi : \mathcal{U} \to \mathbb{R}^n$ is C^1 with bounded derivative on $\mathcal{U}$. (Guaranteed by Lipschitz condition using Lipschitz const.)

<u>Issue</u> Want $\operatorname{diam}(\Phi(C_i^N)) \to 0$ as $N \to \infty$

this means if z_1, z_2 close, $|\Phi(z_1) - \Phi(z_2)| \leq M \cdot \underbrace{|z_1 - z_2|}_{\leq \frac{\sqrt{n}}{2^N}}$.

For ②. quite technical. Suffices to prove for cube centered at origin, by changing coords, shifting function so that $\Phi(0) = 0$.

<u>Require</u> : Φ, Φ^{-1} differentiable with Lipschitz derivative, M Lipschitz ratio for both.

then for every $\epsilon > 0$, $\exists\, \delta > 0$ s.t. if $C \subseteq [-\delta, \delta]^n$,

$$(1-\epsilon)\,[D\Phi(0)] \cdot C \subseteq \Phi(C) \subseteq (1+\epsilon)\,[D\Phi(0)]\, C$$

(δ depends only on M not on anything else about Φ.)

Here $[D\Phi(0)]\, C$ means stretch basis vectors of C by acting by matrix $[D\Phi(0)]$

<u>Rewrite</u> :

$$\Phi(C) - [D\Phi(0)]\, C \subseteq [D\Phi(0)] \cdot \epsilon C$$

$$\Leftrightarrow \quad [D\Phi(0)]^{-1} \left(\Phi(C) - [D\Phi(0)]\, C \right) \subseteq \epsilon C$$

i.e. true for all $x \in C$.

for ③ :

$$\int_Y f(y)\,|d^n y| \approx \sum_{C \in D_N(\mathbb{R}^n)} \sup_{y \in \Phi(C)} f(y)\ \mathrm{vol}_n(\Phi(C))$$

since $\{\Phi(C)\}$ is paving

$$\approx \sum_{C \in D_N(\mathbb{R}^n)} \sup_{x \in C} (f \circ \Phi)(x) \cdot \mathrm{vol}_n(C) \cdot |\det(D\Phi(x_c))|$$

since Jacobian approximates change in vol. at $\underline{x}$ well.

any point $x_c \in C$, say center.

$$\approx \int_X (f \circ \Phi)(\underline{x})\ |\det(D\Phi(\underline{x}))|\ |d^n \underline{x}|$$

definition of integration