How do integrals behave with respect to limits?

Limits are key construction in real analysis. Explore their relation to Riemann integral. Allow us to determine when pull limit inside integral (and thus, for example, take a derivative inside integration)

<u>Q</u> : Under what conditions can we say that

$$\lim_{k \to \infty} \int f_k = \int \left(\lim_{k \to \infty} f_k\right) \quad ? \qquad (*)$$

<u>A</u>. ok if $\{f_k\}$: bounded integrable functions <u>converging uniformly</u> to f, supported in fixed $B \subset \mathbb{R}^n$, ball. Then f integrable and $(*)$ is valid.

Remember – uniform convergence means,

for every $\epsilon > 0$, $\exists N$ s.t. $k \geq N$, then for all $\underline{x} \in \mathbb{R}^n$ (or supp. of functions f_k)

$$|f_k(\underline{x}) - f(\underline{x})| < \epsilon.$$

"uniform" means same <u>N</u> works for every $\underline{x}$. (In weaker definition of pointwise convergence, N can depend on choice of x.)

<u>Example</u> : ① $f_k(x) = \begin{cases} x^k & \text{on } [0,1] \\ 0 & \text{else.} \end{cases}$

f_k converges pointwise to $f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{else.} \end{cases}$

What about uniform convergence? Fails. f not continuous, but if $\{f_k\} \to f$ uniformly, know f is continuous. ↯.

More concretely, given $\epsilon > 0$, like $\epsilon = 1/10$, what should we choose for N? if $x = 1/10$, $N = 2$. , if $x = 1/$[illegible], $N = 4$ if $x = 9/10$, $N = 22$, not too hard to turn this into pf.

⓪ $f_k(x) : [0,1] \to \mathbb{R} = x/k \xrightarrow[\text{pointwise}]{} f(x) = 0$ as $k \to \infty$.

(pick $k \geq \varepsilon/x$)

What about uniformly?

Pick $\epsilon > 0$, find N s.t. if $k \geq N$: want $\left| f_k(x) - f(x) \right| = \frac{x}{k} < \epsilon \quad \forall x \in [0,1]$.

But $\frac{x}{k} < \frac{1}{k} < \frac{1}{N}$ if $k \geq N$

so pick $N > 1/\epsilon$.

Indeed

0 : $\lim_{k \to \infty} \int_0^1 \frac{x}{k} \, dx$

$$= \lim_{k \to \infty} \frac{x^2}{2k} \Big|_0^1 = \lim_{k \to \infty} \frac{1}{2k} = 0 = \int_0^1 \underbrace{\lim_{k \to \infty} \frac{x}{k}}_{0} \, dx. \checkmark$$

1 : $$\lim_{k \to \infty} \int_0^1 x^k \, dx = \lim_{k \to \infty} \frac{x^{k+1}}{k+1} \Big|_0^1 = 0$$

$$= \int_0^1 \underbrace{\lim_{k \to \infty} x^k}_{\substack{0 \text{ everywhere} \\ \text{but } x = 1}} \, dx.$$

STILL OK!

SHOULD BE STRONGER CRITERION.

2 : "moving bump" function

$$f_k(x) = \begin{cases} 1 & \text{if } x \in [k, k+1] \\ 0 & \text{else.} \end{cases}$$

converges pointwise to $f = 0$, but not uniformly. (can never beat $\epsilon < 1$ for all x.)

this fails. $$\lim_{k \to \infty} \int_0^1 f_k = \lim_{k \to \infty} 1 = 1 \neq \int_0^1 \underbrace{\lim f_k}_{=0} = 0.$$

pf of theorem (4.11.2 in book) Given $\epsilon > 0$, since convergence is uniform, we can pick K so that $\sup_{\underline{x} \in \mathbb{R}^n} |f(\underline{x}) - f_k(\underline{x})| < \epsilon$ if $k > K$

then $$L_N(f) > L_N(f_k) - \epsilon \cdot \underbrace{\mathrm{vol}(B)}_{\text{supp. of all } f_k} \quad \text{for any } N$$

$$U_N(f) < U_N(f_k) + \epsilon \, \mathrm{vol}(B) \quad \text{for any } N$$

$$\Rightarrow \quad U_N(f) - L_N(f) < \underbrace{\underbrace{U_N(f_k) - L_N(f_k)}_{\substack{\text{make this small} \\ \text{by choosing } N \text{ large} \\ \text{since all } f_k \\ \text{assumed integrable.}}} + 2\epsilon \, \mathrm{vol}(B)}_{\text{can be made arbitrarily small.}} \quad \text{for any } N.$$

Improvement: "Dominated convergence theorem" (Arzela, 1885)

$\{f_k\}$ integrable. Find R s.t. $\mathrm{supp}(f_k) \subset B_R(0)$ $\forall k$ and such that $|f_k| \leq R$.

If f integrable s.t. $\lim_{k \to \infty} f_k = f$ except on set of measure 0,

then
$$\lim_{k \to \infty} \int_{\mathbb{R}^n} f_k(\underline{x}) \, |d^n \underline{x}| = \int_{\mathbb{R}^n} \lim_{k \to \infty} f_k(\underline{x}) \, |d^n \underline{x}|$$
$$= \int_{\mathbb{R}^n} f(\underline{x}) \, |d^n \underline{x}|$$

this includes case of example 1 from before.

Technical statement, but key to showing Lebesgue integral is well-defined...

Ready to define Lebesgue integral of more general class of functions.

Suppose $f = \sum_{k=1}^{\infty} f_k$ with f_k (Riemann) integrable.

and such that $\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\underline{x})| \, |d^n x| < \infty.$ (*)

this condition assures that $\sum f_k$ converges, but only up to set of measure 0.

so can't really say $\sum f_k = f$, just that, in books' notation

$\sum f_k \underset{L}{=} f$

equal up to set of measure 0.

As long as we only care about integrals of these functions that's ok.

then define Lebesgue integral of f

as $\int_{\mathbb{R}^n} f(\underline{x}) \, |d^n \underline{x}| \overset{\text{DEF}}{:=}$ this sum.

Two results that guarantee Lebesgue integral is well-defined:

① Proposition 4.11.5: If (*) holds

then $\sum f_k$ converges pointwise "almost everywhere"

i.e. except on a set of measure 0.

② Thm 4.11.7 (uses dominated conv. theorem in proof)

if (*) holds for $\sum f_k$, $\sum g_k$ and

$\sum f_k \underset{L}{=} \sum g_k$, then

Lebesgue integrals are equal. i.e.

$$\sum_k \int_{\mathbb{R}^n} f_k(\underline{x}) \, |d^n \underline{x}| = \sum_k \int_{\mathbb{R}^n} g_k(\underline{x}) \, |d^n \underline{x}|$$