Monday Lecture: Given a parametrization $\gamma: \mathcal{U} \to M$; k-manifold in $\mathbb{R}^n$

computing $\mathrm{Vol}_k(M) = \displaystyle\int_{\mathcal{U}} \sqrt{\det(D\gamma^T(u)\, D\gamma(u))}\ |d^k u|$

or insert $f(\gamma(u))$ to do weighted volumes.

Now we want to use this to explore local structure of manifolds.

Basic question. What is volume of "disc" on manifold?

What do we mean by disc?

Take a k-disc inside $\mathcal{U} \subseteq \mathbb{R}^k$, map by γ, depend on parametrization γ. Turn notion around - disc on M is set of points (having some) arc length $\leq r$ from given point.

Q: What is the volume of this dish?

In $\mathbb{R}^1$, just as boring for curves versus real line. 1-volume is $2r$.

i.e. 1-dim'l manifolds

For 2-dim'l manifolds, is $D_r(p) = \pi r^2$?

Or does it depend on how bumpy/curvy the surface S is near p?

Mean this as local calculation, so can assume that, for small enough r, our surface is graph of smooth function of two variables.

For simplicity, call these variables x, y. Think of $S \subset \mathbb{R}^3$

as graph $\begin{pmatrix} x \\ y \\ f\binom{x}{y} \end{pmatrix}$ in $\mathbb{R}^3$ of smooth function f.

To calculate area, need to parametrize points on surface. Use polar.

$$g : \begin{pmatrix} \rho \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ f\begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix} \end{pmatrix}$$

Only written this formally. Difficulty — still need to identify open set $\mathcal{U}_r \subseteq \mathcal{U}$

s.t. $g(\mathcal{U}_r)$ roughly $D_r(p)$.

f is complicated function, but can approximate it using multi-var. Taylor series expansion.

(better since hard to compute f if manifold given by F.

but can compute arbitrarily high Taylor series terms using

chain rule.)

Return to ch. 3, Section 8, to understand Taylor polynomials and what they

tell us about "curviness" of surface.

Start slow with curviness of curves. Work our way up to surfaces.

Given a curve (1-manifold), write it as graph of smooth function locally

$\begin{pmatrix} x \\ f(x) \end{pmatrix}$ then compute the Taylor expansion of f.

May choose coordinates so that origin is point p on curve we're interested in

and so that $f'(0,0) = 0$ (i.e. tangent line is x-axis)

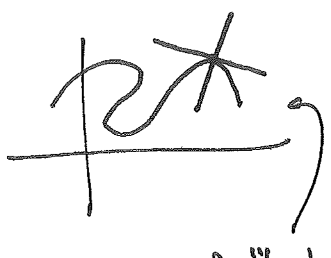

illustration of "best coordinates" for plane curve.

in these "best coordinates", taylor expansion for f

begins: $\frac{a_2}{2}x^2 + \frac{a_3}{3!}x^3 + \cdots$

then define curvature at p on curve $= |a_2|$.

<u>Example</u>: circle of radius r, centered at origin. then at point $\begin{pmatrix} 0 \\ r \end{pmatrix}$, best coordinates are $(X, Y) = (x, y-r)$.

Write $y = \sqrt{r^2 - x^2}$ so $Y = y - r = \sqrt{r^2-x^2} - r = \sqrt{r^2 - X^2} - r$

with Taylor poly. having $f''(0) = \frac{d}{dX^2}\left(\sqrt{r^2-X^2} - r\right)\Big|_{X=0} = \frac{d}{dX}\left(\frac{1}{2}(r^2-X^2)^{-1/2} \cdot -2X\right)\Big|_{X=0}$

$= \frac{d}{dX}\left(\frac{-X}{\sqrt{r^2-X^2}}\right)\Big|_{X=0} = -1 \cdot (r^2 - X^2)^{-1/2} \underbrace{- X \cdot (\cdots)}_{=0 \text{ at } X=0}\Big|_{X=0} = -\frac{1}{r}$.

So curvature of circle is $\boxed{\frac{1}{r}}$

<u>Another approach</u>: Define "best fit circle" to curve using p, two points equidistant to p. 3 pts. uniquely determine a circle. Take circle resulting from limit as distance from $p \to 0$.

then define curvature at $p = \frac{1}{r}$, r: radius of this circle.

Often preferable to have formula for curvature that works without having to use "best coordinates"

Proposition: Given $y = f(x)$ in neighborhood of $\begin{pmatrix} a \\ f(a) \end{pmatrix}$, the curvature at $\begin{pmatrix} a \\ f(a) \end{pmatrix}$ is

$$\frac{|f''(a)|}{(1+|f'(a)|^2)^{3/2}}$$

Do example from book. — $x^3 + xy + y^3 = 3$ implicitly defines 1-manifold.

Compute its derivatives via chain rule. at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, write $\begin{pmatrix} g(y) \\ y \end{pmatrix}$ or as $\begin{pmatrix} x \\ g(x) \end{pmatrix}$ and we find $g'(1) = -1$

How do we find coeffs. of Taylor expansion?

in the latter case

$g''(1) = -5/2$.

$k = 5/4\sqrt{2}$.

Compute Jacobian

$$[3x^2+y,\ 3y^2+x]\Big|_{(x,y)=(1,1)}$$

$$= [4, 4].$$

(map to $\mathbb{R}$, onto, choose pivot column ...)

by chain rule $[Dg] = -\begin{bmatrix} D \text{ from pivot} \\ \text{cols} \end{bmatrix}^{-1} \cdot \begin{bmatrix} D \text{ from non-piv.} \\ \text{columns} \end{bmatrix}$

$= -1$ (regardless of pivot we choose)

To find second order terms, use that

$F\begin{pmatrix} p^{(2)}(1+h) \\ 1+h \end{pmatrix}$ vanishes to order 2. Write $p^{(2)}(1+h)$

$= 1 - h + \frac{1}{2} g''(1) h^2 + \cdots$

Sub in, and each coefficient of h must be identically 0. Find $g''(1)$ by solving for h^2 term $= 0$.