On Monday, discussing volumes of discs on manifolds – particularly surfaces.

Set up a volume integral, but had two problems.

(1) Remember how to find Taylor polynomials at $p \in M$.

(2) Find open set $\mathcal{U}$ to parametrize $D_r(p)$ on M: $\gamma: \mathcal{U} \to D_r(p)$.

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Almost done with (1) – remember how to find 2^{nd} degree terms in Taylor expansion.

along the way, defined curvature of 1-manifold at point p:

if p locally expressible as $\begin{pmatrix} x \\ f(x) \end{pmatrix}$ in $\mathbb{R}^2$, then curvature at p

is $\dfrac{|f''(a)|}{|1+f'(a)^2|^{3/2}}$. (in best coordinates X, Y, it is just $|g''(0)|$ if $Y = g(X)$)

Wanted to move on to surfaces:

Again, we can pick best coordinates at p for 2-manifold in $\mathbb{R}^3$:

locally like $\begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$. Pick new coords X, Y, Z so that p is at $(0,0,0)$ in new coords. and tangent plane to p is X,Y-plane with Z axis normal to this plane.

Then in these best coords, we have near $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} X \\ Y \\ f(X,Y) \end{pmatrix}$ with

Taylor expansion starting in degree 2 again:

write it as $\frac{1}{2}\left(A_{2,0} X^2 + 2A_{1,1} XY + A_{0,2} Y^2 \right)$ + higher order terms

studied these quadratic forms in optimization section. ← wrote as sum of (or diff of) squares, in yet another change of coords.

In these coordinates, we have two notions of curvature (generalizing the one from 1-dimensional manifolds before)

You can think of this quadratic form $A_{2,0}X^2 + 2A_{1,1}XY + A_{0,2}Y^2$

as $$(X \ Y)\underbrace{\begin{pmatrix} \partial^2 f/\partial X^2 & \partial^2 f/\partial Y \partial X \\ \partial^2 f/\partial X \partial Y & \partial^2 f/\partial Y^2 \end{pmatrix}}_{\substack{\text{symmetric matrix, "Hessian"} \\ \text{hence } 2A_{1,1}}}\begin{pmatrix} X \\ Y \end{pmatrix}$$

natural to use basic invariants under conjugation action by invertible matrices: $\det(A)$, $\mathrm{tr}(A)$.

<u>Mean curvature</u> : $\frac{1}{2}\,\mathrm{tr}(A) = \frac{1}{2}(A_{2,0} + A_{0,2}) =: H(p)$

<u>Gauss curvature</u> : $A_{2,0}A_{0,2} - A_{1,1}^2 = \det(A). =: K(p).$

<u>Gauss' theorem</u> : $\mathrm{Area}(D_r(p)) = \pi r^2 - K(p)\frac{\pi}{12} r^4 +$ higher order terms in r.

<u>Problem</u> : How do we compute $K(p)$? Need surface M to be in best coordinates X, Y, Z. Translation part is easy: just subtract.

Now we assume p is at $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$,

$z = f(x,y)$ has Taylor expansion $a_1 x + a_2 y + \frac{1}{2}(a_{2,0}x^2 + 2a_{1,1}xy + a_{0,2}y^2)$ + higher order terms.

We win (i.e. are in best coordinates) if $a_1, a_2 = 0$.

Measure this with positive # $c = \sqrt{a_1^2 + a_2^2}$.

then it turns out that $K(p) = K(\underline{0}) = \dfrac{a_{2,0}\,a_{0,2} - a_{1,1}^2}{(1+c^2)^2}$.

(not so different from the case of plane curves, going from best coords.)

Example: paraboloid $z = x^2 + y^2$. Find (Gaussian) curvature at point (a, b, a^2+b^2)

Write
$X = x - a$
$Y = y - b$
$Z = z - (a^2 + b^2)$

in X, Y, Z coords:

$$Z + a^2 + b^2 = (X+a)^2 + (Y+b)^2$$

$$Z = X^2 + 2aX + 2bY + Y^2$$

$$= 2aX + 2bY + \tfrac{1}{2}(2X^2 + 2Y^2)$$

From this we compute

$c = \sqrt{(2a)^2 + (2b)^2}$

$c^2 = 4a^2 + 4b^2$

$$K\begin{pmatrix} a \\ b \\ a^2+b^2 \end{pmatrix} = \frac{4}{(1 + 4a^2 + 4b^2)^2}$$

e.g. K at origin is 4. this is a max. Away from origin, surface gets flatter.

In general, we can do any function of x, y making similar subtractions.

2nd Example: $x^2 + y^3 + xyz^3 - 3 = 0$ at $(1,1,1)$. $F: \mathbb{R}^3 \to \mathbb{R}$

Jacobian:

$$\left[2x + yz^3,\ 3y^2 + xz^3,\ 3xyz^2\right]\Big|_{(1,1,1)}$$

$3xyz^2\big|_{(1,1,1)} = 3 \neq 0$ so can write z as function of x, y. Call it f

Now solve for $f\begin{pmatrix} 1+X \\ 1+Y \end{pmatrix} - 1$

$a_{1,0} = -1 \qquad a_{0,1} = -4/3$

$a_{2,0} = -2/3 \qquad a_{0,2} = -26/9 \quad , a_{1,1} = -2/3$

so $\quad c = \sqrt{(-1)^2 + (-4/3)^2} = \sqrt{25/9} \quad , \quad c^2 = 25/9.$

$a_{2,0}\, a_{0,2} - a_{1,1}^2 = 52/27 - 4/9 = 40/27.$

$$\frac{40/27}{(1+25/9)^2} = \frac{40/27}{(34/9)^2} = \frac{40}{34^2/3} = \frac{30}{17^2} = .1038\ldots$$