k-forms on $\mathbb{R}^n$ are functions $\phi : \underbrace{(\mathbb{R}^n)^k}_{k \text{ vectors in } \mathbb{R}^n} \longrightarrow \mathbb{R}$, all

linear combinations of the elementary forms $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ ← take $k \times k$ subdeterminants of the $n \times k$ matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix}$

($\wedge$: product taking a k-form and l-form to a $k+l$ - form)

Concluded Wednesday saying the objects we want to integrate are

functions : $\varphi : \mathcal{U} \longrightarrow$ k forms on $\mathbb{R}^n$ $\qquad$ $\mathcal{U}$: open set in $\mathbb{R}^n$

$u \longmapsto \phi_u : (\mathbb{R}^n)^k \to \mathbb{R}$

or, using basis,

$$\sum_{i_1, \dots, i_k} \underbrace{c_{i_1, \dots, i_k}(u)}_{\substack{\uparrow \\ \text{smooth function in applications.}}} dx_{i_1} \wedge \cdots dx_{i_k}$$

<u>Example</u> : $\mathcal{U} \subseteq \mathbb{R}^3$

$\varphi : (x_1, x_2, x_3) \mapsto \phi_{(x_1, x_2, x_3)}$

For example $\underbrace{e^{x_1} \sin x_2}_{\substack{\text{what we called} \\ c_{1,3}(x_1, x_2, x_3) \text{ above.}}} \cdot dx_1 \wedge dx_3$

Now can calculate it on any pair of vectors v_1, v_2 at any $(x_1, x_2, x_3) \in \mathbb{R}^3$

Think of $\phi_{(x_1, x_2, x_3)}(v_1, v_2)$ as volume function on parallelogram spanned by v_1, v_2, based at (x_1, x_2, x_3).

<u>Plan</u> : use these in integrals on oriented manifolds.

problem - still don't know what "orientation" is. Temporary fix ...

Use parametrizations. Parametrizations assign an orientation automatically. See this with unit circle already:

$$\theta \longmapsto \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} \quad \text{parametrizes clockwise}$$

$$\theta \longmapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{parametrizes counterclockwise.}$$

so we expect integration to give arc length 2π or -2π depending on choice of orientation.

Definition: Given a parametrization $\gamma : U \longrightarrow M$: (oriented) manifold

define

$$\int_{[\gamma(U)]} \varphi = \int_U \underbrace{\varphi(u)\,(D\gamma(u))}\,|d^k u|$$

volume of k-//gram with base point u and vectors the column vectors of $D\gamma(u)$

$\gamma : U \subseteq \mathbb{R}^k \longrightarrow M \subseteq \mathbb{R}^n$.

φ is k-form field

Example: $\gamma : U \subseteq \mathbb{R}^2 \longrightarrow M \subseteq \mathbb{R}^2$

that means φ maps to 2-forms on $\mathbb{R}^2$. (only 2-form, up to constant, is determinant.)

$$\varphi(x_1, x_2) \longmapsto f(x_1, x_2)\,\underbrace{dx_1 \wedge dx_2}_{\det}$$

for some smooth f.

Assume f is constant function 1.

then

$$\int_{\gamma(U)} dx_1 \wedge dx_2 = \int_U \det(D\gamma(u))\,|d^k u|$$

just the change of vars. formula but without abs-value signs on the determinant.