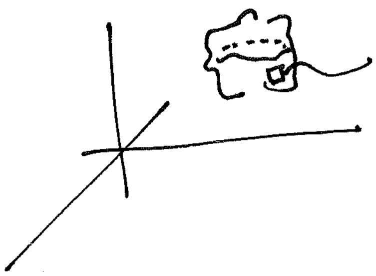
Last week, defined n-dimensional volume of a body $A \subseteq \mathbb{R}^n$ (subset)

as
$$\mathrm{vol}_n(A) = \int_{\mathbb{R}^n} 1_A(\underline{x}) \, |d^n \underline{x}|$$
$$= \int_A 1 \, |d^n \underline{x}|.$$

More interesting applications, attach weight to points in integration domain.

Classic example: density.



density function $\mu(\underline{x})$.

Since mass $=$ volume $\cdot$ density then we define
$$\mathrm{Mass}(A) = \int_A \mu(\underline{x}) \, |d^n \underline{x}|$$

or, weighting by $x_i \, (\mu(\underline{x}))$, obtain center of mass $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)$ with
$$\overline{x}_i \overset{\text{def}}{=} \int_A x_i \cdot \mu(\underline{x}) \, |d^n \underline{x}|.$$

treat object as point mass at its center of mass.

Do something very similar in using integration to calculate probabilities.

<u>The game</u>: think of $\mathbb{R}^n$ as possible outcomes.

want to know the probability that a subset of these $A \subseteq \mathbb{R}^n$ occurs.

<u>example</u>: in $\mathbb{R}^2$, measure an adult male gorilla's height and weight.

height and weight are real numbers. Of course, it won't be negative, so those probabilities are 0.

What is probability that height is 8 ft, weight 800 lbs.? 0.

Same as any other pair of numbers, but probability that it lies in any open set (or set containing open set) is non-zero.

Plan: someone hands you a "magic" probability density function $\mu(\underline{x})$

so that $\text{Prob}(A) = \int_A \mu(\underline{x}) \, |d^n \underline{x}|$.

How did they find this function? We'll do some examples to explore this.

At least we know two things immediately: ① $\mu(\underline{x}) \geq 0 \quad \forall \, \underline{x}$

② $\int_{\mathbb{R}^n} \mu(\underline{x}) \, |d^n \underline{x}| = 1$

(probabilities of events are non-negative. + probability over all possible events is 1.)

Example: coin tosses. Record results as a binary sequence with, say, 1's for heads and 0's for tails. If we think of it as binary equiv. of decimal expansion $.01101\ldots$ then outcomes are points in unit interval $[0,1]$. So $0 = .00\ldots$ = flipping tails forever.

$\frac{1}{2} = .100\ldots$ in binary = flipping one head, then tails forever.

What is probability of landing in $[0, \frac{1}{2})$? First digit 0. (1 tails in 1 flip.)
Probability is $\frac{1}{2}$.

What about in $[0, \frac{1}{4})$? Again 2 tails in 2 flips, $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

For any dyadic interval $\left[\frac{k}{2^N}, \frac{k+1}{2^N}\right)$, this means requirement on first N flips.

So probability is $1/2^N$. i.e. prob. of landing in any dyadic interval is its length

(write k in binary to determine the flips. $k = \sum_{n=0}^{N} a_n 2^n$)

Since all lengths (1 dim'l volumes) are defined by dyadic interval approx., $\text{Prob}(A) = \int_{\mathbb{R}} 1_A(x) \, |d^1 x|$

More typically, given data and you'd approximate the probability density function from it. How would you make such a function?

Height of 10 year old girls. (between 50-60 inches)

How do we calculate average height of 10 yr. old girl? Answer: just like center of mass. Weight the height: $\int_{\mathbb{R}} x\, \mu(x) |dx|$

Fancy lingo: Given function f, if $f \cdot \mu(x)$ is integrable, we call

$$\int_{\mathbb{R}} f(x)\, \mu(x)\, |dx| \doteq E(f) \text{ , the "expectation" of } f.$$

(f = "random variable")

To dig deeper, we might want to understand how much height varies away from mean height (Expectation $E(x)$).

Compute difference and integrate: $E(f - E(f))$? No. This is 0.

Better: $E\left((f - E(f))^2 \right)$ = "Variance" $V(f)$.

More famous: Standard deviation $\sigma(f) := \sqrt{V(f)}$

But these are just definitions. Are they really useful? Answer in terms of the central limit theorem. – if you know $E(f)$, $\sigma(f)$, then you can make predictions: More precisely, perform n trials, resulting in $x_1, \dots, x_n$ with average result $\underline{\overline{x}} = \frac{1}{n}(\underline{x}_1 + \cdots + \underline{x}_n)$

the probability that $\bar{x} \in [a,b]$ is given by:

$$\frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma(f)} \cdot \int_a^b e^{-\frac{n}{2}\left(\frac{x - E(f)}{\sigma(f)}\right)^2} dx$$

normal distribution scaled/spread out by E and σ.

Use it for coin tossing: $E(x)$ in coin-tossing is $1/2$. (each outcome equally likely, or $\int_{\mathbb{R}} x \cdot 1_{[0,1]}\, dx = 1/2$.)

$$\sigma(x) = \sqrt{E((x - 1/2)^2)}$$

$$\sigma^2 = \int_{\mathbb{R}} (x - 1/2)^2\, 1_{[0,1]}\, dx = \frac{1}{3}(x - 1/2)^3 \Big|_0^1$$

$$= \frac{1}{3}\left(1/2^3 - (-1/2)^3\right)$$

$$= 1/12.$$

(if "experiment" is flipping coin infinitely many times)

one coin toss; then need to do sums not integrals. As in example in book.