skipping discussion of non-orientable manifold.

(mention Möbius strip as intuitive example)

<u>Theorem</u> (6.4.10 in H & H)

$M \subseteq \mathbb{R}^n$, a smooth k-manifold. U_1, U_2 open in $\mathbb{R}^k$

with params. $\gamma_1 : U_1 \to M$, $\gamma_2 : U_2 \to M$ orientation preserving.

then $\displaystyle\int_{[\gamma_1(U_1)]} \varphi = \int_{[\gamma_2(U_2)]} \varphi$, φ : k-form field defined on M.

<u>Consequence</u> : We may define $\displaystyle\int_M \varphi$ independent of orientation-preserving parametrization.

<u>pf of theorem</u> : try to relate the two integrals via change of vars theorem.

cut out subsets of U_1, U_2 on which change of vars hypotheses fail (have measure 0)

$\underbrace{U_1 \setminus \underbrace{X_1}_{\text{relaxed param}} \text{ and } \setminus Y_2 = (\gamma_1^{-1} \circ \gamma_2)(X_2)}_{\text{book calls this } U_1^{ok}}$

NOTE: PESKY ABSOLUTE VALUE!

consider $\Phi = \gamma_2^{-1} \circ \gamma_1 : U_1^{ok} \to U_2^{ok}$ use change of vars formula here b/c Φ is nice: C^1, Lipschitz inverse, ...

$$\int_{[\gamma_2(U_2)]} \varphi = \int_{U_2} \varphi(\gamma_2(u_2)) \left[D\gamma_2(u_2)\right] |d^k u_2| \underset{\text{CH OF VARS}}{=} \int_{U_1} \varphi(\gamma_2 \circ \Phi(u_1)) \underbrace{D(\gamma_2 \circ \Phi)(u_1)}_{D\gamma_2(\Phi(u_1))} \underbrace{|d^k u_1|}_{|\det D\Phi(u_1)|}$$

Lemma (Prop. 6.4.8 in H-H):

$\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathbb{R}^k$ $\quad \gamma_1 : \mathcal{U}_1 \to M$, $\gamma_2 : \mathcal{U}_2 \to M$ relaxed params.

γ_1, γ_2 are both orientation preserving or both orientation reversing

if and only if $\forall$ $u_1 \in \mathcal{U}_1^{ok}$, $u_2 \in \mathcal{U}_2^{ok}$, paired up

s.t. $\gamma_1(u_1) = \gamma_2(u_2)$,

$$\det\left[D(\gamma_2^{-1} \circ \gamma_1)(u_1) \right] > 0.$$

this guarantees that det appearing will be positive,

so pesky absolute value is inconsequential

if parametrizations are <u>orientation preserving</u>

Physical intuition behind our favorite form fields in $\mathbb{R}^3$.

0-forms: 0-manifold is disconnected set of points.

0-form is function taking no vectors, returning elt. of $\mathbb{R}$.

fancy way to say 0-form is a number.

0-form field – assigns number to each point $x \in M$, 0-Manifold (aka function on M.)

1-forms: Previously, constructed 1-forms for orientations using:

$\underline{x} \in M$, $\underline{v} \in T_x M$ then $(\underline{x}, \underline{v}) \longmapsto \underline{w}(\underline{x}) \cdot \underline{v}$

for some vector $\underline{w}$ depending on $\underline{x}$.

If we expand $\underline{w} \cdot \underline{v}$ into elementary 1-forms

get $w_1 dx_1 + \cdots + w_n dx_n$. Better:

$$\sum_{i=1}^{n} w_i(\underline{x})\, dx_i$$

How many 1-form fields look like this? ALL OF THEM!

Thinking of them as linear combinations of dx_i good for computing, etc.

as $\underline{w}(\underline{x}) \cdot v$ good for physical intuition.

$\underline{w}(x)$: vector field in $\mathbb{R}^n$, $\underline{w}(x) \cdot v$ work done by vector field along direction of v.

"work form for the vector field $\underline{w}(\underline{x})$"

per unit of length.

(draw pictures, intuition from gravity, etc.)

2-forms: Favorite way to pick 2-form in $\mathbb{R}^3$ - find vector off of tangent space $n(\underline{x})$, compute: $\det [n(\underline{x}), v_1, v_2]$

with v_1, v_2 basis for $T_{\underline{x}} M$.

$$\det \begin{pmatrix} n_1(\underline{x}) & | & | \\ n_2(\underline{x}) & v_1 & v_2 \\ n_3(\underline{x}) & | & | \end{pmatrix} = n_1(\underline{x}) \cdot dy \wedge dz - n_2 \, dx \wedge dz + n_3(\underline{x}) \, dx \wedge dy.$$

expanding along first column

Which 2-forms look like this? ALL of them! (same true for $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ not just $\mathbb{R}^2 \subset \mathbb{R}^3$)

Physical intuition: $n(\underline{x})$ as field of vectors.

(velocity vector for moving fluid)

$\det (\underline{n}(\underline{x}) \; v_1 \; v_2)$ measures volume of fluid flowing through parallelogram spanned by v_1, v_2 per unit time.

(examine cases of ~~parallel to~~ field and perpendicular to field)

3-form: Only one. $\det (v_1 \; v_2 \; v_3) = dx \wedge dy \wedge dz$. Weight this volume by computing volume of a chunk of space. density function f then get "mass" (described this long ago for probability, then again in the context of manifolds)