On Wednesday, we gave a criterion for integrability - two forms: f always bounded w/ bounded support.

Version 1 (if and only if) f integrable iff for every

$\epsilon > 0$, $\exists\ N \gg 0$ s.t. $\displaystyle\sum_{\substack{\text{cubes } C_{k,N} \text{ with} \\ \operatorname{osc}_{C_{k,N}}(f) > \epsilon}} \operatorname{vol}(C_{k,N}) < \epsilon$.

Version 2 If f continuous except on set of volume 0,

then f integrable.

(idea: where f continuous, uniformly continuous, so can choose $N \gg 0$ so that no cubes have big oscillation. (i.e. $> \epsilon$). Then show points of discontinuity can be covered by cubes of total vol. $< \epsilon$)

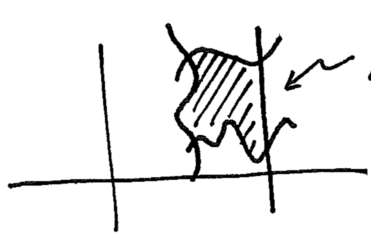
Now - given continuous function f, region of $\mathbb{R}^n \supseteq A$,

can we compute $\displaystyle\int_A f\,|d^n x|$? Well $\displaystyle\int_{\mathbb{R}^n} f \cdot 1_A\,|d^n x|$

is a continuous function except possibly on ∂A. Just need to ensure ∂A has volume 0.

Cor 4.3.8 in book: If ∂A is finite union of graphs of functions, then $\operatorname{vol}(\partial A) = 0$.

e.g.



$A \subseteq \mathbb{R}^2$.

Section 4.4 — give improved form of Version 2 which is "if and only if"

New concept : "measure 0" (applies more generally than volume 0)

For set X : volume 0 means , for every $\epsilon > 0$, $\exists\ N \gg 0$ s.t.

$$\sum_{\substack{C_{k,N}: \\ C_{k,N} \cap X \neq \phi}} \mathrm{vol}(C_{k,N}) \le \epsilon .$$

measure 0 means for every $\epsilon > 0$, cover X with ~~[illegible]~~ cubes B_i in $\mathbb{R}^n$ (possibly infinite)

s.t. $\sum_{B_i} \mathrm{vol}_n(B_i) \le \epsilon$. (Note $\mathrm{vol}(B_i) \to 0$ as $i \to \infty$ in order to converge)

(in measure 0, cubes need not be dyadic. Any length, any starting point)

For example, rationals in $[0,1]$ are not pavable, i.e. no volume, because upper, lower Riemann sums always give 1, 0 resp.

But it does have measure 0 — often repeated trick : enumerate the rationals : $0, 1, 1/2, 1/3, 2/3, \ldots$

call them $a_1, a_2, \ldots$. Given $\epsilon > 0$:

put "cube" around a_i of length $\frac{\epsilon}{2^i}$

then $\sum \mathrm{vol}(\text{cubes}) = \epsilon$

using same trick, we can show if $\{X_i\}$ sets of measure 0 (countably many)

then $\bigcup_i X_i$ has measure 0 .

Big theorem : f bounded with bounded support $: \mathbb{R}^n \to \mathbb{R}$, f is integrable if and only if f is continuous except on set of measure 0 .

proof starts out as before:

just need to show that condition in version 1 is satisfied:

For every $\epsilon > 0$, $\exists$ $N >> 0$ s.t. $\displaystyle\sum_{\substack{C_{K,N}: \\ \operatorname{osc}_{C_{K,N}}(f) > \epsilon}} \operatorname{vol}(C_{K,N}) < \epsilon$. (**)

on continuous part of f, set is empty as before. The discontinuous points D have measure 0, so contained in $\bigcup_i B_i$, boxes with $\sum_i \operatorname{vol}(B_i) < \epsilon$

so suffices to show $\exists$ $N >> 0$ s.t. if

$\operatorname{osc}_{C_{K,N}}(f) > \epsilon$, then $C_{K,N} \subset$ [illegible] $\bigcup_i B_i$

Suppose not. Then ~~$\exists$~~, for each N, $\exists$ x_N, y_N, z_N ~~[illegible]~~ some

$|f(x_N) - f(y_N)| > \epsilon$, $z_N \notin \bigcup B_i$.

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then (**) $\leq$ ~~[illegible]~~

$\operatorname{vol}(\bigcup_i B_i)$

$\leq \sum_i \operatorname{vol}(B_i)$

just one sneaky problem!

~~[illegible]~~ $\bigcup_i B_i$ may not be pavable.

if B_i's were rational #'s (closed box of length 0) this would be false.

Fix: Show B_i s.t. $\operatorname{osc}_{B_i}(f) > \epsilon$ are finite in number.

proof: Prune set of boxes B_i so that none are fully contained in another. Otherwise useless.

If claim above is false, then $\exists$ ∞-ly many i's with $\operatorname{osc}_{B_i}(f) > \epsilon$, make two infinite sequences $\{x_i\}$, $\{y_i\}$ s.t. $x_i, y_i \in B_i$ with $|f(x_i) - f(y_i)| > \epsilon$.

These x_i's, y_i's are in bounded set, so have convergent subsequence. B_i's have volume $\to 0$ so must converge to same point p.

Moreover p is a point of discontinuity since, even as x_i, y_i get closer,

$|f(x_i) - f(y_i)| > \epsilon$, the fixed ϵ we chose at the beginning.

This means $p \in$ one of the B_i (as they cover set of discontinuities)

call this box B_p. But then as $i \gg 0$, $B_i \subset B_p$

since $x_i, y_i \in B_i$, also close to p.

this contradicts fact that and $\mathrm{vol}(B_i) \to 0$.

B_i were pruned initially. //