Fubini's theorem : Most important result in integral calculus for computing integrals. First, give statement for $\mathbb{R}^2$. $\underline{x} = (x_1, x_2)$

$f(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$ integrable. How to compute

$$\int_{\mathbb{R}^2} f(x_1,x_2) \, |d^2(x_1,x_2)| \quad ? \qquad (*)$$

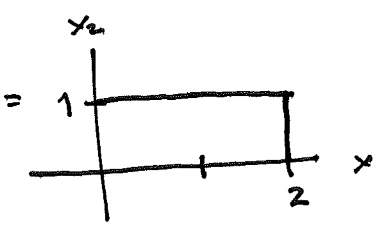
Ans : if for each $x_1 \in \mathbb{R}$, the function $x_2 \mapsto f(x_1,x_2)$ is integrable, then

$$(*) = \int_{\mathbb{R}} \underbrace{\left(\int_{\mathbb{R}} f(x_1,x_2) \, |dx_2| \right)}_{\substack{\text{one variable integral in} \\ \text{variable } x_2 \text{, treating} \\ x_1 \text{ as constant.} \\ \text{Can be handled by FTC.}}} |dx_1|$$

"iterated integral"

(In particular $x_1 \mapsto \int f(x_1,x_2)|dx_2|$ is integrable)

For example, $\displaystyle\int_{\mathbb{R}^2} 1_{\underset{R}{\text{Rectangle}}} |d^2x|$ with $R =$



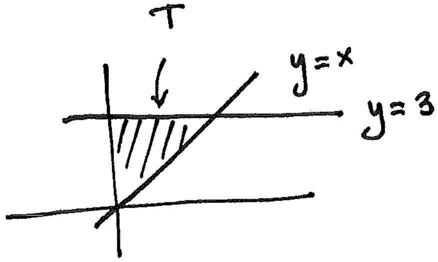
(we know the answer, and could even compute it with Riemann sums, but let's see what Fubini's Thm. says)

Is the map, for fixed x_1, of form $x_2 \mapsto 1_R(x_1,x_2)$ integrable?

Yes, because for each $x_1 \in (0,2)$, the map is $x_2 \mapsto 1_{(0,1)}(x_2)$, which is integrable.

So get $\displaystyle\int_{\mathbb{R}} \int_{\mathbb{R}} 1_R(x_1,x_2) \, |dx_2| \, |dx_1| = \int_0^2 \int_{\mathbb{R}} 1_{(0,1)}(x_2)|dx_2| \, |dx_1|$

Second example :



$$\int_T x^2 y \; |d(x,y)|$$

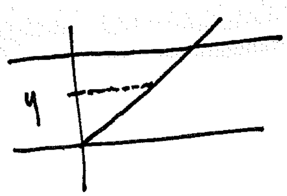
Fix x, ask if $y \mapsto x^2 \cdot y \cdot 1_T$ is integrable. Yes! Now, if $x \in (0,3)$, then $y \in (x,3)$ so this can be written

$$\int_0^3 \underbrace{\int_x^3 x^2 y \, dy}_{= \, x^2 \cdot \left(\frac{1}{2} y^2\right)\Big|_x^3 \, = \, \frac{9}{2}x^2 - \frac{1}{2}x^4} dx$$

now integrate this from 0, 3 in x to finish.

$$= \frac{3}{2}x^3 - \frac{1}{10}x^5 \Big|_0^3 = \frac{81}{2} - \frac{243}{10} = 3^4 \underbrace{\left(\frac{1}{2} - \frac{3}{10}\right)}_{1/5}$$

Also could have integrated in x first, fixing value of y.

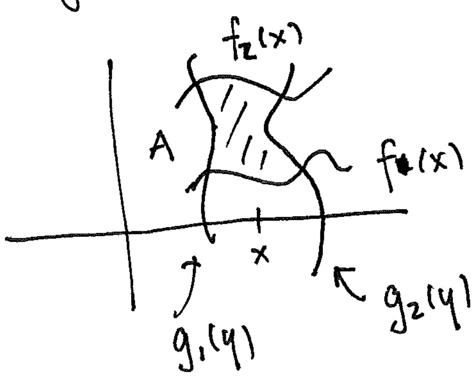


$$\int_0^3 \int_0^y x^2 y \, dx \, dy$$

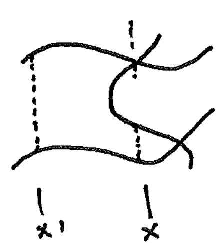
$$= \int_0^3 \underbrace{\frac{1}{3}x^3 \cdot y \Big|_0^y}_{y^4/3} dy = \frac{1}{15} y^5 \Big|_0^3 = 81/5. \checkmark$$

In general in $\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} 1_A \, |d(x,y)| \overset{(?)}{=} \int_{g_1(y)}^{g_2(y)} \int_{f_1(x)}^{f_2(x)} \bullet \, |dy|\,|dx|$$



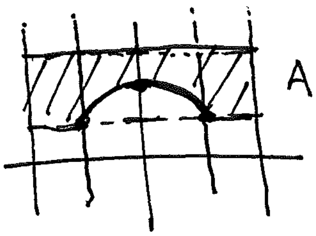
No! Much more complicated.



← may need to break up integral into pieces to solve it.

Concrete example : domain in $\mathbb{R}^2$ bounded by lines

$y = 1, \quad y = 3, \quad x = -2, \quad x = 2, \quad y = 2 - x^2$



A

then do $\displaystyle\int_{\mathbb{R}^2} \cancel{A}\, 1_A \, |d(x,y)|$

$$= \int \int \quad = \int_{-2}^{-1} \int_1^3 1 \, dy \, dx$$

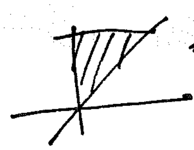
x ~~first~~ outside

y inside

$$+ \int_{-1}^{1} \int_{2-x^2}^{3} 1 \, dy \, dx$$

$$+ \int_{1}^{2} \int_{1}^{3} 1 \, dy \, dx$$

Sometimes it really pays off to do one integration before another. Book's contrived example here is e^{-y^2} over domain



$\triangle \; T = \{ (x,y) \mid 0 \le x \le y \le 1 \}$

One way: $\displaystyle\int_0^1 \int_x^1 e^{-y^2} \, dy \, dx$ and it is impossible! No elementary function with derivative e^{-y^2}.

Second way: $\displaystyle\int_0^1 \int_0^y e^{-y^2} \, dx \, dy$. Saved since inner integration gives exactly the needed substitution factor for outer integration in y.

General statement of Fubini's theorem:

if $y \mapsto f(\underline{x}, y)$ is integrable then
$\mathbb{R}^m \to \mathbb{R}$

$$\int_{\mathbb{R}^{n+m}} f(\underline{x}, \underline{y}) \, |d^{n+m}(\underline{x}, \underline{y})| = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(\underline{x}, \underline{y}) \, |d^m \underline{y}| \, |d^n \underline{x}|$$