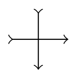
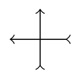
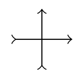
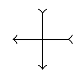
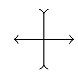
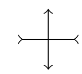
Math 8300 – Quantum Groups – Problem Set 1
Due: Friday, February 21

1. (Experimenting with the 6-vertex model) In this exercise, we use the weights from our example in class:

j 	j 	j 	j 	j 	j 
1	1	λ_j	λ_j	$1 - q\lambda_j$	$1 - q^{-1}\lambda_j$

where λ_j's are parameters attached to row j in the model, and $q \neq 0$.

a) Give an example of an admissible filling of the six-vertex model (you choose the number of rows and columns, but don't make it too big) and find the Boltzmann weight of the state, defined to be the product of the Boltzmann weights at each vertex.

b) We assign an orientation to the arrows, so that up and left arrows are $+$ and down and right arrows are $-$. Choose an ordering on the basis elements of $V \otimes V$ where V has basis $\{v_+, v_-\}$ and find a 4×4 matrix $R(\lambda_j; q)$ in $\mathrm{End}(V \otimes V)$ so that its entries encode the Boltzmann weights above.

c) Use your answer in (b) to verify an example of the identity $T = R_{01}R_{02} \cdots R_{0N}$ where T is the one-row partition function ("transfer matrix") in $\mathrm{End}(V_{(0)} \otimes V^{\otimes N})$ and $R_{0,j}$ denotes the matrix R applied to the 0-th and j-th factors in $V_{(0)} \otimes V^{\otimes N}$ and the identity on every other factor. Does your example suggest a general method of proof, and if so, can you sketch it?

d) Write down one of the required identities for a solution of the Quantum Yang-Baxter equation (QYBE) with weights $R(\lambda_1; q)$ and $R(\lambda_2, q)$, by making a particular choice of 6 boundary arrows. Verify that your equations are satisfied by the solution R to the QYBE given in Theorem 1 of `arXiv:0912.0911`.

2. (The Train Argument) In class, we saw that with toroidal boundary conditions, the QYBE implies that transfer matrices commute. Explain what can go wrong when the left-hand and right-hand boundaries are not toroidal, and what the QYBE implies in those cases.

3. (Tensor products of vector spaces) We took the definition of $V_1 \otimes V_2$ for granted. This exercise helps build familiarity with this construction. Recall $V_1 \otimes V_2$, as vector spaces over a field k, are just finite sums of elements in $V_1 \times V_2$ modulo the relations:

$$(\lambda v_1, v_2) = (v_1, \lambda v_2),\ (v_1, v_2) + (v_1', v_2) = (v_1 + v_1', v_2),\ (v_1, v_2) + (v_1, v_2') = (v_1, v_2 + v_2'),$$

for all v_1, v_1' in V_1, v_2, v_2' in V_2 and λ in k.

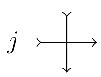
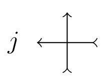
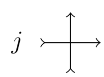
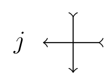
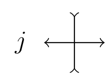
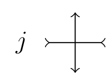
a) Show $k \otimes V \simeq V \simeq V \otimes k$ as vector spaces, for any vector space V.

b) Let $V^* = \text{Linear}_k(V, k)$, the vector space of k-linear maps from V to k. Show they indeed form a vector space for which there is map realizing $V \subseteq (V^*)^*$. Also show that $V_1^* \otimes V_2^* \subseteq (V_1 \otimes V_2)^*$. Are each of these inclusions actually isomorphisms? Why or why not?

4. (Counting ASMs)

a) Show that there is a bijection between $N \times N$ alternating sign matrices (ASMs) and admissible states (i.e., fillings) of the $N \times N$ square lattice six-vertex model with domain wall boundary conditions (in arrows on sides, out arrows on top and bottom).

In Stroganov and Okada's proof of the ASM conjecture, the above Boltzmann weights are used, where $x = \beta_j/\alpha_i$ if the vertex appears in row i and column j of

					
$qx - q^{-1}x^{-1}$	$qx - q^{-1}x^{-1}$	$x - x^{-1}$	$x - x^{-1}$	$q - q^{-1}$	$q - q^{-1}$

the matrix. Here $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N$ are arbitrary non-zero parameters. (It doesn't really matter, but for definiteness for your dear grader (me), you could number rows and columns in ascending order from left to right and bottom to top, so bottom-leftmost vertex has parameters (α_1, β_1).)

b) Show that the partition function Z_N on the $N \times N$ domain-wall boundary lattice with these weights, when multiplied by α_i^{N-1} (resp. β_i^{N-1}), is a polynomial in α_i^2 (resp. β_i^2) of degree at most $N - 1$.

c) Show that Z_N obeys the following recursion relation when $\alpha_1 = \beta_1$:

$$Z_N(\alpha_1, \ldots, \alpha_N; \beta_1, \ldots, \beta_N) =$$
$$(q - q^{-1}) \prod_{i=2}^{n} (q\alpha_1/\alpha_i - q^{-1}\alpha_i/\alpha_1) \prod_{j=2}^{n} (q\beta_j/\alpha_1 - q^{-1}\alpha_1/\beta_j) Z_{N-1}(\alpha_2, \ldots, \alpha_N; \beta_2, \ldots, \beta_N)$$

d) Show that the previous two properties, the fact that $Z_1 = q - q^{-1}$, and the symmetry of Z_N in the α's and β's (separately, which follows from the QYBE) uniquely determine Z_N.