On Monday, ended with "working definition" of quantum groups as non-commutative, non-cocommutative Hopf algebras – Hopf: bialgebra with antipode.

bialgebra : Given H-modules M_1, M_2 (mean: as algebra modules) or even more precisely – left-algebra modules.

then $M_1 \otimes M_2$ has left H-module structure

via $\Delta(h) \cdot (m_1 \otimes m_2) \underset{\text{Sweed.}}{=} \sum h_{(1)} \cdot m_1 \otimes h_{(2)} \cdot m_2$

antipode : $S : H \to H$ playing role of inverse (anti-algebra, anti-coalgebra map)

though not necessarily $S^2 = id$.

and cocommutative H have $M_1 \otimes M_2 \simeq M_2 \otimes M_1$

in general, they may be different.

Recall that if M is an (left) H-module, then we can define left H-module structure

in $M^* = \mathrm{Hom}_k(M, k)$ ← linear functionals on M

via $(h \cdot f)(m) \stackrel{\text{def}}{=} f(S(h)m)$.

[For generic algebra A, with left A-module M then M^* comes with structure of a right A-module]

Both of these will lead to important consequences when we explore repns/modules of Hopf algebras.

Also, if S invertible (not guaranteed in defn of Hopf alg.)

then can also define another left H-module:

$$(h \cdot f)(m) \stackrel{\text{def}}{=} f(S^{-1}(h)m)$$

(if $S^2 = id$ then $S = S^{-1}$, so they are the same. But might be different in general.)

if H is finite dim'l then antipode is bijective (so invertible)

[existence of (both) duals : "rigid monoidal category"]

Give additional examples of Hopf algebras to improve our intuition...

$(q \neq 0 \in k)$

Proposition / Example: $U_q(b_+)$: algebra generated by ~~$1, x, g, g^{-1}$~~ with relations

$$\cancel{gg^{-1} = 1 = g^{-1}g}, \quad \cancel{gX = qXg}$$

make this g, g^{-1} into H, H^{-1}: too close to q in typeface...

$1, X, H, H^{-1}$ with $HH^{-1} = 1 = H^{-1}H$, $HX = qXH$.

not commutative!

Then the maps $\Delta X = X \otimes 1 + H \otimes X$

$\Delta H = H \otimes H$ $\qquad \varepsilon X = 0$

$\Delta H^{-1} = H^{-1} \otimes H^{-1}$ $\qquad \varepsilon H = 1 = \varepsilon H^{-1}$

$S(X) = -H^{-1}X$

$S(H) = H^{-1}$

$S(H^{-1}) = H$

makes $U_q(b_+)$ into an infinite dimensional non-comm. non-cocomm. Hopf algebra with S invertible, but $S^2(X) = q^{-1}X$.

(may be extended to)

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pf: Extend def'n of Δ, ε on generators multiplicatively, check that they are algebra maps (in particular, consistent with relations in algebra)

For example, want to show extending Δ mult. gives $\Delta(HX) = \Delta(qXH)$:

$$\Delta(HX) \underset{\substack{\uparrow \\ \text{extend} \\ \text{mult.}}}{=} \Delta(H)\Delta(X) = (H \otimes H)(X \otimes 1 + H \otimes X)$$

$$= HX \otimes H + H^2 \otimes HX$$

while

$$\Delta(qXH) \underset{\substack{\uparrow \\ \text{extend} \\ \text{mult.} \\ \text{(and linearly)}}}{=} q\,\Delta(X)\Delta(H) = q(X \otimes 1 + H \otimes X)(H \otimes H)$$

$$= q(XH \otimes H) + q \cdot (H^2 \otimes XH)$$

$$\underset{\substack{\uparrow \\ \text{using rel'n} \\ HX = qXH}}{=} HX \otimes H + H^2 \otimes HX \ \checkmark$$

then must follow them on products since Δ, ε extended mult.

etc. As before, want to extend antipode S as antialgebra map. Then check antipode axioms on generators.

check it is not cocommutative: Immediate on generator X.

$$\begin{tikzcd} B \arrow[d, "\Delta"'] \arrow[dr, "\Delta"] & \\ B\otimes B \arrow[r, "\tau"] & B \otimes B \end{tikzcd} \qquad \begin{array}{c} \Delta(X) = X\otimes 1 + H \otimes X \\ \big\downarrow \tau \\ 1 \otimes X + X \otimes H \neq \Delta(X). \end{array}$$

Also, what is dimension of $U_q(b_+)$? Not finite dim'l since $\{ H^a X^b \mid a \in \mathbb{Z}, b \in \mathbb{Z}_{\geq 0} \}$ are linearly independent.

$$S^2(X) = S(S(X)) = S(-H^{-1}X) = -S(X)S(H^{-1})$$
$$= +H^{-1}\underbrace{XH}_{q^{-1}HX} = q^{-1}X$$

more importantly, $S^2(u) = H^{-1}uH$ $\forall\ u \in U_q(b_+)$ since

S^2 is homom. of $U_q(b_+)$, and this shows S^2 bijective, so S bijective, hence invertible.

— [insert remark about b_+ as Lie alg. of Borel s.g. of $SL(2,\mathbb{C})$]

Dual Hopf algebras. In finite-dim'l case, dual vector space H^* has same dimension as H and $H^{**} \cong H$.

so we tend to think of H, H^* symmetrically, write $\langle \phi, v \rangle := \phi(v)$ for "evaluation map" to reflect this symmetry.

then we can give formulas for H^* as Hopf algebra using Hopf alg. structure on H:

$$\langle \phi \cdot \psi, a \rangle := \langle \phi \otimes \psi, \Delta a \rangle$$
$$\langle 1, a \rangle := \varepsilon(a)$$
$$\langle \Delta\phi, a \otimes b \rangle := \langle \phi, ab \rangle$$
$$\varepsilon(\phi) := \langle \phi, 1 \rangle$$
$$\langle S\phi, a \rangle = \langle \phi, Sa \rangle$$

axioms are symmetric upon reversing arrows + interchanging roles of (m, η) and (Δ, ε).

Here, works because $(H \otimes H)^* \cong H^* \otimes H^*$ are equalities of v.s. in f.dim'l case.

More generally, say H, H' are "dually paired" if $\exists \ \langle \, , \rangle : H' \otimes H \to k$

satisfying previous 5 axioms.

(H : finite dim'l, then evaluation map is unique such $\langle \, , \rangle$. Infinite dim'l is more subtle.

In fact, there can be more than one choice for H'.)

Examples : kG, $\mathcal{U}(\mathfrak{g})$ or $T(V)$, now $\mathcal{U}_q(b_+)$. Find dually paired Hopf algebras for them.

(1)

if G finite, then we know.

$(kG)^*$: algebra of functions on G — call it $k(G)$ with values in k, pointwise products : $(f \cdot g)(x) \stackrel{def}{=} f(x)g(x)$

$\forall x \in G, \ f, g \in k(G)$.

so coproduct

$$(\Delta f)(x, y) = f(xy) \quad \text{identifying } k(G) \otimes k(G) = \underbrace{k(G \times G)}_{\text{functions on two group vars.}}$$

counit $\varepsilon(f) := f(e)$ $\quad$ e : id. in G

antipode. $Sf(x) = f(x^{-1})$ $\quad$ using earlier $\langle \, , \rangle$ properties.

(2) $\mathcal{U}(\mathfrak{g})$'s dually paired Hopf algebra — Recall that $\mathfrak{g}$: finite dimensional cx semisimple Lie algebra

$\xrightarrow{\text{exponential map}} G \subset \mathrm{Mat}_n(\mathbb{C})$ for some n, a complex Lie group. In fact, G is complex alg. variety,

so G can be cut out of $\mathrm{Mat}_n(\mathbb{C})$ by polynomial equations $\{ p(x) = 0 \}_{x \in \mathrm{Mat}_n(\mathbb{C})}$

Write $\mathbb{C}[G]$ for coordinate algebra of this algebraic variety.

$$= \mathbb{C}[x_{i,j}]_{i,j=1}^{n} \Big/ \{ p(x) = 0 \}$$

matrix coordinates

Example 2 cont. : Matrix mult. suggests natural def'n of comult. / counit on $\mathbb{C}[G]$.

$$\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j} \qquad \varepsilon(x_{i,j}) = \underbrace{\delta_{i,j}}_{\text{Kronecker } \delta}$$

$= 0$ if $i \neq j$
$= 1$ if $i = j$

also can define over k: $k[G]$ since structure constants are integers,

so then $\mathbb{Z}[G]$ is well-defined and then we can tensor with k.

Antipode is more complicated in terms of $\underbrace{\text{cofactors}}$ in $\{x_{i,j}\}$.

removing one row and one column.

so $k[SL_2]$ has generators $\{a, b, c, d\}$ then $S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

slight abuse of notation...