Last time, we met our first non-comm, non-cocomm. Hopf algebra $U_q(b_+)$ $q \neq 0 \in k$.

$$= \langle 1, X, K, K^{-1} \mid KK^{-1} = K^{-1}K = 1,\ KX = qXK \rangle$$

K, K^{-1} grouplike: Δ: diag., S: inv., ε: maps to 1

$$\Delta(X) = X \otimes 1 + K \otimes X, \quad S(X) = -K^{-1}X, \quad \varepsilon(X) = 0.$$

(indicated how to extend these maps, check compatibility w/ algebra relations)
MORAL: this is possible ∫ once make defn on generators...

Also explained notion of dual Hopf algebra. H, H' are in duality if $\exists$ (non-degenerate) pairing $\langle\, ,\, \rangle : H' \otimes H \to k$ satisfying 5 axioms:

① $\langle \phi \cdot \psi, a \rangle = \langle \phi \otimes \psi, \Delta a \rangle$

② $\langle 1, a \rangle = \varepsilon(a)$

③ $\langle \Delta\phi, a \otimes b \rangle = \langle \phi, ab \rangle$

④ $\varepsilon(\phi) = \langle \phi, 1 \rangle$

⑤ $\langle S\phi, a \rangle = \langle \phi, Sa \rangle$

Think: H is given, and then axioms realize Hopf structure on H'.

(Takeuchi, '81)

Examples: ① kG : gp. algebra, $k(G)$: algebra of functions on G. $\langle\, ,\, \rangle$: evaluation map $\langle f, v \rangle := f(v)$.
(G finite)

② $U(\mathfrak{g})$: univ env. alg. of $\mathfrak{g}$: finite dim'l cx. semisimple alg. is dually paired with $\mathbb{C}[G]$ G: Lie gp assoc. to $\mathfrak{g}$ via exp. map. and $\mathbb{C}[G]$ is its coordinate algebra:

$G \subseteq Mat_n(\mathbb{C})$ for some n (e.g. $SL_n \subset Mat_n(\mathbb{C})$), cut out by polynomial equations $p(\underline{x})$ in the matrix coordinates $x_{i,j}$. And $\mathbb{C}[G]$ is algebra

$$:= \mathbb{C}[x_{i,j}]_{i,j=1}^{n} \Big/ \{ p(\underline{x}) \}$$

the coalgebra structure on $\mathbb{C}[G]$ is suggested by matrix mult.:

$$\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j}, \qquad \varepsilon(x_{i,j}) = \delta_{i,j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}.$$

(all this works over k (char 0), since structure consts. defined over $\mathbb{Z}$, then $\otimes k$.)

antipode map is more complicated in terms of cofactors of $\{x_{i,j}\}$ (remove row and column)

e.g. $k[SL_2]$ has gens $\{a,b,c,d\}$ and use suggestive matrix notation and write $S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

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dual pairing determined by values on $\alpha \in \mathfrak{g}$: $\langle \alpha, x_{i,j} \rangle = \rho(\alpha)_{i,j}$ $\quad$ ρ: defining rep'n of $\mathfrak{g} \hookrightarrow \mathrm{Mat}_n(\mathbb{C})$

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see Kassel, Quantum Groups, V.7, for a proof. (~5 pages)

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(3) $U_q(b_+)$ is self-dual. Before we explore this, why is duality important?

Why is duality important?

- sets up duality of (left-) modules for H and (right-) comodules for H' : $\beta: M \to M \otimes H'$, $m \mapsto m^{(1)} \otimes m^{(2)}$ (in H')
- there are several constructions of $U_q(sl_2)$ (and more generally, $U_q(\mathfrak{g})$) using duality.
  - action of $SL_q(2)$ on quantum plane, $U_q(sl_2)$ is its dual.
  - Quantum double construction of Drinfeld : $H, H' \rightsquigarrow D(H,H')$ which, for finite dim'l Hopf algebras, guarantees that $D(H,H')$ is a quasi-triangular Hopf algebra
    has abstract QYBE.

Two facts about $\mathcal{U}_q(b_+)$:

Fact 1: $\mathcal{U}_q(b_+)$ has basis $\{X^m K^n\}_{\substack{m \geq 0 \\ n \in \mathbb{Z}}}$ / k.

pf: Show they span by checking that monomials are stable under mult. by any elt. in $\mathcal{U}_q(b_+) =: \mathcal{U}$, which we can check on generators. If so, then $\mathrm{Span}\{X^m K^n\}_{\substack{m \geq 0 \\ n \in \mathbb{Z}}} \supseteq \mathcal{U} \cdot 1 = \mathcal{U}$. This is short exercise in using relations:

e.g. $K \cdot X^m K^n = q^m X^m K^{n+1}$, etc.

For linear independence, consider (comm.) ring $R = k[A, B, B^{-1}]$ with basis $A^m B^n$ $m \geq 0, n \in \mathbb{Z}$

with endoms: $f(A^m B^n) = A^{m+1} B^n$

$g(A^m B^n) = q^{m} A^m B^{n+1}$ $\quad g^{-1}(A^m B^n) = q^{-m} A^m B^{n-1}$

$g \circ f(A^m B^n) = g(A^{m+1} B^n) = q^{m+1} A^{m+1} B^{n+1}$ $\quad$ so $g \circ f = q \cdot f \circ g$

$f \circ g(A^m B^n) = f(q^m A^m B^{n+1}) = q^m A^{m+1} B^{n+1}$

so have map $\mathcal{U} \longrightarrow \mathrm{End}_k(R)$ and $f, g^{\pm}$ linearly indep in $\mathrm{End}_k(R)$ (supported on diff. mons. in A, B) so $X, K^{\pm 1}$ must be too

$X \longmapsto f$

$K^{\pm 1} \longmapsto g^{\pm 1}$

e.g. $f^m g^n(1) = A^m B^n$

Fact 2: $$\Delta(X^m) = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q X^{m-r} K^r \otimes X^r$$

where $\begin{bmatrix} m \\ r \end{bmatrix}_q = \dfrac{[m]_q!}{[r]_q!\,[m-r]_q!}$ and $[r]_q! = [r]_q [r-1]_q \cdots [1]_q$

and $[r_q] = 1 + q + \cdots + q^{r-1} = \dfrac{1-q^r}{1-q}$

(so that $[r_q] \to r$ as $q \to 1$)

convention: $\begin{bmatrix} m \\ m \end{bmatrix}_q = \begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1.$

pf of Fact 2 : Δ defined by extending multiplicatively, and

$\Delta(X) = X \otimes 1 + K \otimes X$. Apply q-binomial formula to summands:

$$(A+B)^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q A^m B^{n-m} \quad \text{if} \quad qAB = BA.$$

(prove by induction)