Last week, we were in the middle of the proof that $U_q(b_+)$ is self dual,

using 5 axioms in terms of inner products.

Had shown two facts: <u>Fact 1</u>: $U_q(b_+)$ has basis $\{X^m K^n\}_{\substack{m\geq 0\\ n\in\mathbb{Z}}}$ $/k$

<u>Fact 2</u>: $\Delta(X^m) = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q X^{m-r} K^r \otimes X^r$

wanted to explain briefly how to finish pf. from there, following sketch in

Lecture 2 of Majid, QG primer (Proposition 2.5):

The equations (identities) $\langle K, K\rangle = q$, $\langle X, X\rangle = 1$, $\langle X, K\rangle = \langle K, X\rangle = 0$

uniquely determine pairing (and we may then check axioms on basis elts.)

<u>pf</u>. For any $m\geq 0$, $n\in\mathbb{Z}$, let $f_{m,n}(u) := \langle X^m K^n, u\rangle$.

Given the basis, it suffices to show that the axioms for pairing determine formula for all such $f_{m,n}$. Start with its evaluation on generators:

first $f_{m,n}(K) = \langle X^m K^n, K\rangle \overset{\cdot}{=} \langle X^m \otimes K^n, \Delta K\rangle$

pairing axiom: mult. in H' $\longleftrightarrow$ comult in H

$\overset{\cdot}{=} \langle X^m \otimes K^n, K\otimes K\rangle$

defn of ΔK

$= \langle X^m, K\rangle \langle K^n, K\rangle$

bilinear. form extends pairwise to tensors

$= q^n \cdot \delta_{m,0}$.

Even here, need to be careful about $\langle K^n, K\rangle$ if $n<0$. Works out.

$f_{m,n}(X) = \langle X^m K^n, X\rangle =$

$= \langle X^m \otimes K^n, \Delta X\rangle$

$\overset{\cdot}{=} \langle X^m \otimes K^n, X\otimes 1\rangle + \langle X^m \otimes K^n, K\otimes X\rangle$

bilinear, defn of ΔX

$= \langle X^m, X\rangle \langle K^n, 1\rangle + \langle X^m, K\rangle\langle K^n, X\rangle$

so $= \langle X^m, X\rangle \underbrace{\varepsilon(K)^n}_{=1}$

Now
$\langle 1, X\rangle = \varepsilon(X) = 0$
$\langle K, X\rangle = 0$ from defn.
$\langle K^2, X\rangle = \langle K\otimes K, X\otimes 1 + K\otimes X\rangle = 0, \ldots$

and $\langle X^m, X\rangle$: $m=2$: $\langle X^2, X\rangle = \langle X\otimes X, \Delta X\rangle$

$= \langle X\otimes X, X\otimes 1 + K\otimes X\rangle$

$= \langle X, X\rangle \underbrace{\varepsilon(X)}_{0} + \underbrace{\langle X, K\rangle}_{0} \langle X, X\rangle$

$= 0.$

so $\langle X^m, X\rangle = \delta_{m,1}$, and

hence $f_{m,n}(X) = \delta_{m,1}$.

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finally given u, u', want formula for $f_{m,n}(u\cdot u')$:

$$f_{m,n}(uu') = \langle X^m K^n, uu'\rangle = \langle \Delta(X^m K^n), u\otimes u'\rangle$$

$$\overset{\text{extended } \Delta\text{- multiplic.}}{=} \langle \underbrace{\Delta(X^m)}_{\text{binomial expr.}} \underbrace{\Delta(K^n)}_{K^n\otimes K^n}, u\otimes u'\rangle = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q f_{m-r, n+r}(u) \cdot f_{r,n}(u')$$

so indeed find formula for $f_{m,n}$, and so $\langle\ ,\ \rangle$ is uniquely determined. //

New adjective – "quasitriangular"

recall that modules for a bialgebra formed "monoidal category" (Eilenberg), or sometimes "tensor category" – often monoidal op. (like here) is a tensor product on objects.

want this to be associative. $(A,B) \to A \otimes B$.

(family of isomorphisms $a_{A,B,C}$
from $(A\otimes B)\otimes C \to A \otimes (B \otimes C)$
with compatibility constraints for A, B, C, D).

Our potential issue – $A \otimes B$ may be different from $B \otimes A$.
if our bialgebra H is cocommutative, then τ gives natural isom.
for all A, B ($\tau : a\otimes b \mapsto b \otimes a$)

Weaken this. ask for a natural family of isomorphisms $c_{A,B} : A \otimes B \to B \otimes A$
with compatibility with associativity:

$$\begin{tikzcd}
& A\otimes(B\otimes C) \arrow[r, "c_{A,B\otimes C}"] & (B\otimes C)\otimes A \arrow[dr, "a_{B,C,A}"] & \\
(A\otimes B)\otimes C \arrow[ur, "a_{A,B,C}"] \arrow[dr, "c_{A,B}\otimes 1"'] & & & B\otimes(C\otimes A) \\
& (B\otimes A)\otimes C \arrow[r, "a_{B,A,C}"] & B\otimes(A\otimes C) \arrow[ur, "1\otimes c_{A,C}"'] &
\end{tikzcd}$$

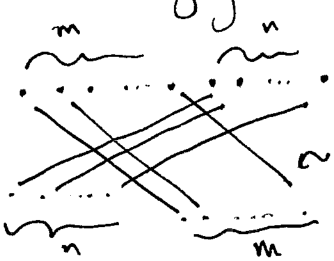
+ companion hexagon for $c_{AB,C}$

"braided monoidal category" – prime example: braids on n-strings (morphisms), $n \in \mathbb{N}$ (objects) $0, 1, 2, \ldots$

maps: multiplication: stacking of braids (comp. of morphisms)

tensor product: putting braids next to each other.

$c_{m,n}$



~ m strings over n strings.

What are possible "braidings" in $\mathrm{Mod}_k(H)$: H-modules ?
H : bialgebra.

Given braiding $c_{A,B} : A \otimes B \to B \otimes A$, then H is a module, and so $\exists$ $c_{H,H} : H \otimes H \to H \otimes H$ with a distinguished elt.

$$\gamma = c_{H,H}(1 \otimes 1) \in H \otimes H.$$

Conversely, given any elt $\gamma = \sum_i u_i \otimes v_i \in H \otimes H$, then get morphism

$$c_{A,B} : A \otimes B \to B \otimes A$$
$$a \otimes b \mapsto \sum_i (u_i \cdot b) \otimes (v_i \cdot a)$$

and this is a bijection.

we want $c_{A,B}$ to be a module morphism, an isomorphism, and to satisfy the associativity compatibility. (so $c \cdot (h \cdot (a \otimes b)) = h \cdot c(a \otimes b)$)

Set $\gamma =: R \in H \otimes H$, get following translation...

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bijection above seen from the following comm. diagram where for each $m \in M$, $\hat{m} : H \to M$: module takes $\hat{m}(1) \Rightarrow m$

$$\begin{tikzcd}
k \otimes k \arrow[r, "\eta \otimes \eta"] & H \otimes H \arrow[r, "c_{H,H}"] \arrow[d, "\hat{m} \otimes \hat{n}"'] & H \otimes H \arrow[d, "\hat{n} \otimes \hat{m}"] \\
& M \otimes N \arrow[r, "c_{M,N}"] & N \otimes M
\end{tikzcd}$$

A $\underline{\text{quasitriangular}}$ Hopf algebra is a pair (H, R) where

H is a Hopf algebra and $R \in H \otimes H$ is an invertible elt. s.t.

Reference: Lecture 5 of Majid, QG Primer

$$\tau \circ \Delta h = R (\Delta h) R^{-1} \quad \forall\, h \in H. \quad \longleftarrow \text{product in } H\otimes H \text{ on both sides.}$$

(so $R = id$ would be "cocommutative" condition) and s.t.

in $H_{(1)} \otimes H_{(2)} \otimes H_{(3)}$ we have the following identities:

$(*)$ $$(\Delta \otimes id) R = R_{13} R_{23} \qquad (id \otimes \Delta) R = R_{13} R_{12}$$

(where, as before R_{13} means " ~~[illegible]~~)
if $R = R^{(a)} \otimes R^{(b)}$, then R_{13} means $R^{(a)} \otimes 1 \otimes R^{(b)}$

$\underline{\text{Lemma}}$: If (H, R) is quasitriangular, then

① $(\varepsilon \otimes id) R = (id \otimes \varepsilon) R = 1$, $(S \otimes id) R = R^{-1}$
$(id \otimes S) R^{-1} = R$

② $(H, \tau(R^{-1}))$ is quasitriangular

③ $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$ in $H \otimes H \otimes H$. (abstract QYBE)

Pf. for ①, have $(\Delta \otimes id) R = R_{13} R_{23}$ from axioms,

$(*)$ means $(\Delta \otimes id)(R^{(a)} \otimes R^{(b)}) = \Delta(R^{(a)}) \otimes R^{(b)}$.
$$= R^{(a)}_{(1)} \otimes R^{(a)}_{(2)} \otimes R^{(b)}$$

if
$R = R^{(a)} \otimes R^{(b)}$
$R^{(a)}, R^{(b)} \in H$ (in general; might be linear combination of them...)

Apply $\varepsilon \otimes id \otimes id$, $\quad \varepsilon(R^{(a)}_{(1)}) \underbrace{R^{(a)}_{(2)}}_{R^{(a)}} \otimes R^{(b)}$

coalgebra axiom $\longrightarrow$

$$\begin{tikzcd} C\otimes C \arrow[d, "\varepsilon\otimes id"'] & \\ k\otimes C \arrow[r, phantom, "\cong"] & C \arrow[ul, "\Delta"'] \end{tikzcd}$$