

## LECTURE 13 Intro

In our last lecture, we saw that weakening cocommutativity from condition

$A \otimes B \xrightarrow{\sim} B \otimes A$   
 $\tau : a \otimes b \mapsto b \otimes a$

to any isomorphisms  $c_{A,B}$  compatible (natural family of) with associativity, module action ("braided monoidal category")

$\Leftrightarrow$  giving invertible elt.  $R \in H \otimes H$  satisfying certain properties.

Restate defn and lemma and prove lemma.

Reminder about  $R$  - "universal  $R$  matrix"

given  $\rho$ : repn of  $H$ , then  $(\rho \otimes \rho)(R)$  defines elt. in  $V \otimes V$  giving solution to QYBE on  $V \otimes V \otimes V$ .

more generally, we can choose three reps  $\rho_1, \rho_2, \rho_3$  with spaces  $(U, V, W)$ .  
get QYBE on  $U \otimes V \otimes W$ .

A quasitriangular Hopf algebra is a pair  $(H, R)$  where

$H$  is a Hopf algebra and  $R \in H \otimes H$  is an invertible elt. s.t.

Reference: Lecture 5 of Majid, QG Primer

$\tau \circ \Delta h = R(\Delta h)R^{-1} \quad \forall h \in H.$  ← product in  $H \otimes H$  on both sides.

(so  $R = id$  would be "cocommutative" condition) and s.t.

in  $H_{(1)} \otimes H_{(2)} \otimes H_{(3)}$  we have the following identities:

(\*)  $(\Delta \otimes id)R = R_{13}R_{23} \quad (id \otimes \Delta)R = R_{13}R_{12}$

(where, as before  $R_{13}$  means ~~the product of  $R$  in the 1st and 3rd components~~)  
if  $R = R^{(a)} \otimes R^{(b)}$ , then  $R_{13}$  means  $R^{(a)} \otimes 1 \otimes R^{(b)}$

Lemma: If  $(H, R)$  is quasitriangular, then

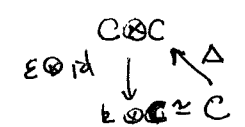
- ①  $(\epsilon \otimes id)R = (id \otimes \epsilon)R = 1, \quad (S \otimes id)R = R^{-1}$   
 $(id \otimes S)R^{-1} = R$
- ②  $(H, \tau(R^{-1}))$  is quasitriangular
- ③  $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$  in  $H \otimes H \otimes H.$  (abstract QYBE)

pf. For ①, have  $(\Delta \otimes id)R = R_{13}R_{23}$  from axioms,

(\*) means  $(\Delta \otimes id)(R^{(a)} \otimes R^{(b)}) = \Delta(R^{(a)}) \otimes R^{(b)}$   
 $= R^{(a)}_{(1)} \otimes R^{(a)}_{(2)} \otimes R^{(b)}$

if  $R = R^{(a)} \otimes R^{(b)}$   
 $R^{(a)}, R^{(b)} \in H$  (in general; might be linear combination of them...)

Apply  $\epsilon \otimes id \otimes id$ ,  $\epsilon(R^{(a)}_{(1)})R^{(a)}_{(2)} \otimes R^{(b)}$   
coalgebra axiom  $\rightarrow R^{(a)}$



For ①,  $(\Delta \otimes id) R = R_{13} R_{23}$  from axioms,

and  $(\epsilon \otimes id) \Delta = id$  (coalgebra axiom  $\begin{matrix} C \otimes C \\ \epsilon \otimes id \downarrow \quad \uparrow \Delta \\ k \otimes C \simeq C \end{matrix}$ )

So  $((\epsilon \otimes id) \otimes id) (\Delta \otimes id) R = R$

and on the other hand  $= (\epsilon \otimes id \otimes id) R_{13} R_{23}$   
 $= (\epsilon \otimes id) R \underbrace{\epsilon(1)}_1 R$   
*E is alg. map.*

since  $R$  invertible, then comparing two sides,  $(\epsilon \otimes id) R = 1$  as desired.

use other coalgebra axiom for  $id \otimes \epsilon$  to prove

$(id \otimes \epsilon) R = 1$  in same fashion.

② is straightforward since  $\tau$  is simple map, so checking axioms easy.

③

$R_{12} R_{13} R_{23} \stackrel{\text{axiom}}{=} R_{12} (\Delta \otimes id) R = (\tau \circ \Delta \otimes id) (R) R_{12}^{ax} = (*)$

now  $R_{12} \Delta(x) R_{12}^{-1} = \tau \circ \Delta(x) \quad \forall x \in H$

i.e.  $R_{12} \Delta(R^{(a)}) = \tau \circ \Delta(R^{(a)}) \cdot R_{12}^{ax}$

$(*) = (\tau \otimes id) (\Delta \otimes id) (R) R_{12}$

$= (\tau \otimes id) (R_{13} R_{23}) R_{12}$

$= R_{23} R_{13} R_{12}$

NOTE THE PARENTHESES!