LECTURE 13 Intro

In our last lecture, we saw that weakening cocommutativity condition from

$A \otimes B \xrightarrow{\sim} B \otimes A$
$\tau : a \otimes b \longmapsto b \otimes a$

to any (natural family of) isomorphisms $c_{A,B}$ compatible with associativity, module action ("braided monoidal category")

$\Longleftrightarrow$ giving invertible elt. $R \in H \otimes H$ satisfying certain properties.

Restate defn and lemma and prove lemma.

Reminder about R - "universal R matrix"

given ρ : repn of H with space V, then $(\rho \otimes \rho)(R)$ defines elt. in $V \otimes V$ giving solution to QYBE on $V \otimes V \otimes V$.

more generally, we can choose three repns ρ_1, ρ_2, ρ_3 with spaces (U, V, W).

get QYBE on $U \otimes V \otimes W$.

A quasitriangular Hopf algebra is a pair (H, R) where

H is a Hopf algebra and $R \in H \otimes H$ is an invertible elt. s.t.

Reference: Lecture 5 of Majid, QG Primer

$$\tau \circ \Delta h = R (\Delta h) R^{-1} \quad \forall h \in H. \quad \leftarrow \text{product in } H \otimes H \text{ on both sides.}$$

(so $R = \mathrm{id}$ would be "cocommutative" condition) and s.t.

in $H_{(1)} \otimes H_{(2)} \otimes H_{(3)}$ we have the following identities:

$$(*) \qquad (\Delta \otimes \mathrm{id}) R = R_{13} R_{23} \qquad (\mathrm{id} \otimes \Delta) R = R_{13} R_{12}$$

(where, as before R_{13} means "[illegible])
if $R = R^{(a)} \otimes R^{(b)}$, then R_{13} means $R^{(a)} \otimes 1 \otimes R^{(b)}$

—

Lemma: If (H, R) is quasitriangular, then

① $(\varepsilon \otimes \mathrm{id}) R = (\mathrm{id} \otimes \varepsilon) R = 1$, $(S \otimes \mathrm{id}) R = R^{-1}$
$(\mathrm{id} \otimes S) R^{-1} = R$

② $(H, \tau(R^{-1}))$ is quasitriangular

③ $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$ in $H \otimes H \otimes H$. (abstract QYBE)

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Pf. For ①, have $(\Delta \otimes \mathrm{id}) R = R_{13} R_{23}$ from axioms,

$(*)$ means $(\Delta \otimes \mathrm{id})(R^{(a)} \otimes R^{(b)}) = \Delta(R^{(a)}) \otimes R^{(b)}$

$= R^{(a)}_{(1)} \otimes R^{(a)}_{(2)} \otimes R^{(b)}$

if

$R = R^{(a)} \otimes R^{(b)}$

$R^{(a)}, R^{(b)} \in H$ (in general, might be linear combination of them...)

Apply $\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}$, $\quad \underbrace{\varepsilon(R^{(a)}_{(1)}) R^{(a)}_{(2)}}_{R^{(a)}} \otimes R^{(b)}$

coalgebra axiom $\longrightarrow$

$C \otimes C$; $\varepsilon \otimes \mathrm{id} \downarrow$, $\nwarrow \Delta$; $k \otimes C \cong C$

For ①, $(\Delta \otimes \mathrm{id}) R = R_{13} R_{23}$ from axioms,

and $(\varepsilon \otimes \mathrm{id}) \Delta = \mathrm{id}$ (coalgebra axiom $C \otimes C \xrightarrow{\varepsilon \otimes \mathrm{id}} k \otimes C \simeq C \xrightarrow{\Delta} C \otimes C$)

So $((\varepsilon \otimes \mathrm{id}) \otimes \mathrm{id})(\Delta \otimes \mathrm{id}) R$

$= R$

and on the other hand $= (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}) R_{13} R_{23}$

$= (\varepsilon \otimes \mathrm{id}) R \ \underbrace{\varepsilon(1)}_{1} R$ ← ε is alg. map.

since R invertible, then comparing two sides, $(\varepsilon \otimes \mathrm{id}) R = 1$ as desired.

use other coalgebra axiom for $\mathrm{id} \otimes \varepsilon$ to prove $(\mathrm{id} \otimes \varepsilon) R = 1$ in same fashion.

— ② is straightforward since τ is simple map, so checking axioms easy.

③

$$R_{12} R_{13} R_{23} \underset{\text{axiom}}{=} R_{12} (\Delta \otimes \mathrm{id}) R = (\tau \circ \Delta \otimes \mathrm{id})(R)\, R_{12} = (*)$$

so ↑

now $R_{12} \Delta (x) R_{12}^{-1} = \tau \circ \Delta (x) \quad \forall x \in H$

i.e. $R_{12} \Delta (R^{(a)}) = \tau \circ \Delta (R^{(a)}) \cdot R_{12}$

$$\begin{aligned}(*) &= (\tau \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R)\, R_{12} \\ &= (\tau \otimes \mathrm{id})(R_{13} R_{23})\, R_{12} \\ &= R_{23} R_{13} R_{12}. \quad \checkmark\end{aligned}$$

NOTE THE PARENTHESES!