Last time we showed that the finite simple modules of U_q := $U_q(sl_2)$

q not a rt. of 1, char$(k) \neq 2$, are indexed by $\pm q^n$:
highest wts.

Called them $L(n, \pm) \simeq M(\pm q^n) / M'$ ← unique submodule; spanned by m_i with $i > n$.

Universal module w/ h.w. $\pm q^n$: $U_q / U_q E + U_q(K \mp q^n)$ with gens. $m_i = [F^i]$

in these modules, nice action: (basis $m_0, \dots, m_n$) ~~[illegible]~~, in $L(n, \pm)$:

$$K \cdot m_i = \pm q^{n-2i} m_i$$

$$F \cdot m_i = \begin{cases} m_{i+1} & \text{if } i < n. \\ 0 & \text{if } i = n \end{cases}$$

$$E \cdot m_i = \begin{cases} \pm [i]_q [n+1-i]_q m_{i-1} & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases}$$

Plan: - introduce quantum Casimir elt. in center of $U_q(sl_2)$.

- show that when q not rt. of unity, these detect fin.dim'l simple modules. (act on non-isom. simple modules by distinct scalars)

- use this to prove, when q is not a rt. of unity, that fin-dim'l U_q-mods are semisimple.

What could go wrong?

If M not simple, then it contains, by def'n, a proper submodule N.

have submodule N, quotient module $L = M/N$. Sit in exact sequence:

$$0 \to N \to M \to L \to 0$$

May not be able to conclude $M = N \oplus L$, sequence may not split!

What if L not simple? then find some N_1 with $N_1/N \cong L_1$, some L_1

What if N not simple? then find some submodule N_2 of N.

Might have chain of the form $M \supset N_1 \supset N \supset N_2 \supset 0$.

Keep refining until we get all successive quotients simple. This is "composition series" for module: $M = M_0 \supset M_1 \supset \dots \supset M_s$ with M_i/M_{i+1} simple.

"chain of length s"

Jordan-Hölder thm: If $M = M_0 \supset M_1 \supset \dots \supset M_s$
$= N_0 \supset N_1 \supset \dots \supset N_t$

are two composition series, the lengths are equal ($s=t$) and factor mods in series are isomorphic.

mention that new notion if not semisimple: indecomposable modules: no direct sum decomp.

splitting: $0 \to N \to M \to L \to 0$ s.t. comp. is identity on L

classic non-splitting ex: $0 \to \underset{2\mathbb{Z}}{\mathbb{Z}} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

Introduce quantum Casimir elt.

$$C := FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} = EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}$$

Lemma: (1) C is central in U_q (check on generators, as usual)

(2) C acts on $M(\lambda)$ by scalar mult. by $(\lambda q + \lambda^{-1}q^{-1})/(q-q^{-1})^2$

(recall basis $m_i = F^i m_0$, and C commutes with F, so C acts by common scalar on all basis vectors. check on m_0.)

(3) C acts on $M(\lambda)$ and $M(\mu)$ by the same scalar only if $\lambda = \mu$ or $\lambda = \mu^{-1} q^{-2}$

(Part (2) + algebra)

Corollary: If q not a root of unity, then if L, L' fin. dim'l modules, C acts by scalar on them, and if same scalar for L, L', then $L \cong L'$.

pf: C acts by scalars on homomorphic image of $M(\lambda)$ (i.e. those $L(n, \pm)$)

Plugging in $\pm q^a, \pm q^b$ into (2) we find either e-values match or q is rt. of unity. //

Theorem: Suppose q not a rt. of unity. $\mathrm{char}(k) \neq 2$. Then any fin. dim'l U_q module M is semisimple

pf: Let $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ be a composition series. Since M_i / M_{i-1} simple, C acts by a scalar μ_i. So $\prod_{i=1}^{r} (C - \mu_i)$ annihilates M. Thus M is direct sum of generalized eigenspaces for C:

$$M = \bigoplus_{\mu} M_{(\mu)} \quad \text{with} \quad M_{(\mu)} = \{ m \in M \mid (C-\mu)^s \cdot m = 0 \text{ for } s \gg 0 \}$$

and each $M_{(\mu)}$ is a submodule of M (since C is central in U_q).

so suffices to prove each of $M_{(\mu)}$ are semisimple. Assume $M = M_{(\mu)}$ from now on.

C acts by multiplication by μ_i on M_i/M_{i-1}, but $(C-\mu)^s$ annihilates M for $s \gg 0$.

start over with more specific M, redo composition series.

so $\mu = \mu_i$ for all i, so $M_i/M_{i-1} \cong L(n, \varepsilon)$ (ε: some ± 1)

if we have composition series of length r, so we bet that $M = \bigoplus^r L(n,\varepsilon)$

Now redivide M according to weight spaces. – action of K.

then $M = \bigoplus_{\nu} M_{\nu}$ ν: weights (not in parentheses!) action of K!

and dimensions respect composition series: $\dim M_\nu = \dim N_\nu + \dim (M/N)_\nu$

$$\dim M_\nu = \sum_{i=1}^{r} \dim (M_i/M_{i-1})_\nu = r \cdot \underbrace{\dim L(n,\varepsilon)_\nu}_{=1}$$

combining with earlier results, $\dim M_\lambda = r$ for the h.w. space

with $\lambda = \varepsilon \cdot q^n$, $n \geq 0$

($Ev = 0$ for $v \in M_\lambda$, so $U_q v \cong L(n,\varepsilon)$)

so r-dim'l space of h.w. vectors with same wt. ~~compare dimensions to~~

~~see M = direct sum of r copies of $L(n,\varepsilon)$.~~ Call them $v_1, \dots, v_r$.

$M = \sum_{i=1}^{r} U v_i$ because $\left(M / \sum_{i=1}^{r} U v_i\right)_\lambda = 0$ and each comp. factor of this quotient module would still be $\cong L(n,\varepsilon)$

and then comparing dimensions, we see sum must be direct.