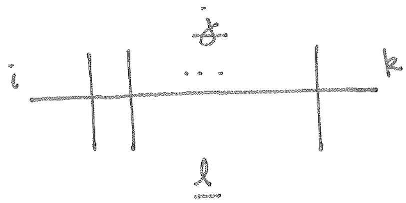
Last time, highlighted role of one-row systems ("transfer matrices") T



with entries $T_{i,j}^{k,l}$

in understanding partition function Z. (e.g. toroidal boundary conditions

$$Z = \mathrm{Trace}_{V^{\otimes N}}\left(\left(\mathrm{trace}_V(T)\right)^M\right) \quad)$$

where $T \in \mathrm{End}(V \otimes V^{\otimes N})$

Do this analysis, by focusing on particularities of given system.

In case of 6-vertex model with toroidal boundary conds., analyzing $\underbrace{\mathrm{Trace}_V(T)}_{\substack{\shortparallel \\ A}} \in \mathrm{End}(V^{\otimes N})$

"transfer matrix"

Done by Lieb + Sutherland (1967)

Alluded to this methodology at end of last time –

use path model of transfer matrices, A breaks up into blocks according to # of paths above + below row. Use method (Bethe Ansatz) to solve for all e-values (distinct) and their e-vectors.

Result: Write $A = P A_{diag} P^{-1}$

P: matrix of eigenvectors

A_{diag}: evalues (distinct)

(in special case where 6 weights are symmetric upon reversing arrows, so have 3 parameters "field-free case")

In particular, turns out P independent of a parameter. Call it x_i in row i.

Since $A_{diag}(x_1)$ commutes with $A_{diag}(x_2)$ then $A(x_1)$ and $A(x_2)$

commute: $A(x_1)A(x_2) = A(x_2)A(x_1)$ "transfer matrices commute"

Q: Is this reasoning reversible?

A: Amazingly, it is reversible and applies to arbitrary boundary conditions.

In Section 9.5 of Baxter's book, he gives list of 6 sufficient conditions for determining e-values of general 6-vertex model transfer matrix :

Steps (i) + (ii) : Prove transfer matrices commute.

Step (iii) : Show transfer matrices are entire functions of cx. parameter.

etc.

So question becomes — Is there a way to ensure (i.e. provide sufficient conds. for...) parametrized transfer matrices commute?

Example SE/NW : 1 NE/SW : λ NS/EW : $1-q\lambda / 1-q^{-1}\lambda$

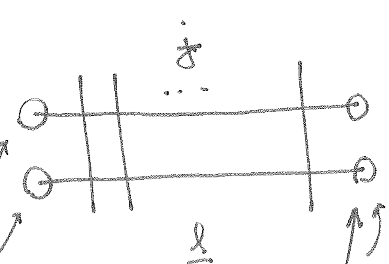
λ : arb. cx. param.
q : $\neq 0$ cx. param.

think : λ : parameter in question.

Proposition : $A(\lambda_1)A(\lambda_2) = A(\lambda_2)A(\lambda_1)$ for any choice of λ_1, λ_2.

proof strategy : pictorial.

For any choice of j, l

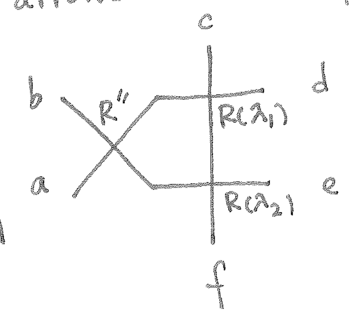


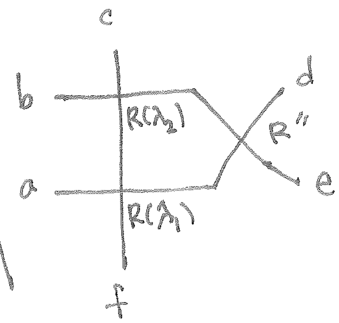
$\leftarrow A(\lambda_1)$ vs. $\leftarrow A(\lambda_2)$
$\leftarrow A(\lambda_2)$ $\quad$ $\leftarrow A(\lambda_1)$

for simplicity : fix some choice here, or consider them internal.

Brilliant idea (Yang, Baxter) :
Swapping of transfer matrices follows from mini-identity of partition functions :

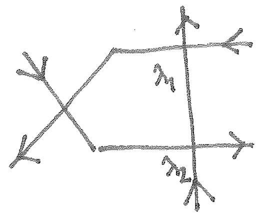
Yang-Baxter equation : Seek a solution R'' : set of weights encoded in matrix so that for any choice of arrows a, b, c, d, e, f :

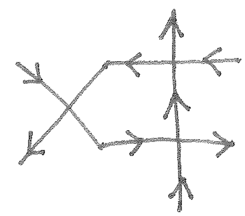
Z (

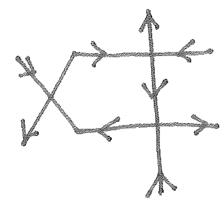
) $=$ Z (

)

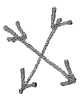
Example for choice of a, b, c, d, e, f.

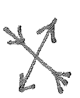
Z (

) $=$ wt () +

wt ()

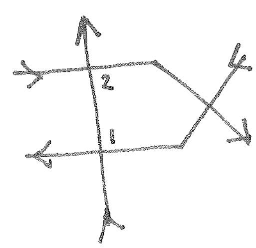
this is left-hand side of YBE. for this choice of a, b, c, d, e, f.

$=$ R'' () $\cdot$ $SE(\lambda_1) \cdot SW(\lambda_2)$

$+$ R'' () $EW(\lambda_1)\, NS(\lambda_2)$

2^6 cases $= 64$. $1/2$ empty.

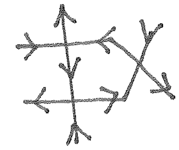
32 cases… 32 equations "quadratic in λ"

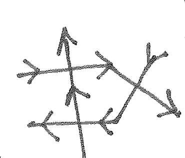
RHS $=$ Z (

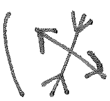
)

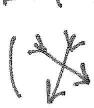
note same boundary conditions

GOAL : Find R'' weights in 6-vertex model that solve all 32 cases simultaneously.

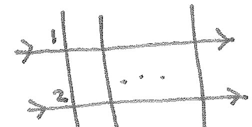
$=$ wt ()

$+$ wt ()

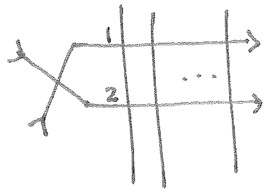
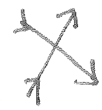
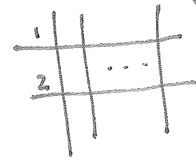
$=$ R'' () $\cdot$ $EW(\lambda_2)\, NS(\lambda_1)$

$+$ R'' () $SW(\lambda_2)\, SE(\lambda_1)$

If we have a solution R'' to YBE, how does it prove commutativity of transfer matrices?

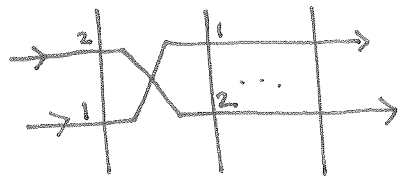
Pictorial proof. Start with $Z\Bigg($  $\Bigg)$

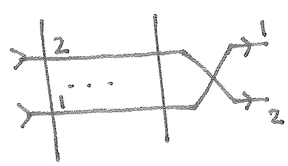
Step
Claim 1.

$Z\Bigg($  $\Bigg) = \mathrm{wt}\Big($  $\Big) \cdot Z\Bigg($  $\Bigg)$

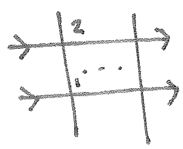
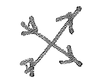
Step 2 ⟶ ||

Use YBE to conclude...

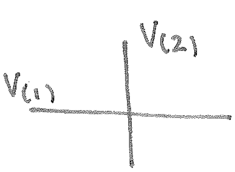
$Z\Bigg($  $\Bigg)$ $\overset{\text{YBE}}{\underset{\text{again}}{=}} \cdots \overset{\text{YBE}}{\underset{\text{again}}{=}}$

$= Z\Bigg($  $\Bigg)$

so these transfer matrices commute!

$= Z\Bigg($  $\Bigg) \mathrm{wt}\Big($  $\Big)$

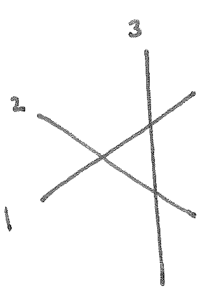
Algebraic interpretation



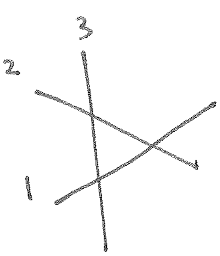
$R \in \mathrm{End}(V_{(1)} \otimes V_{(2)})$

vertical strand – one copy of V

horizontal strand – other copy of V.

In 

3 strands.

⟵ coeff. in $\mathrm{End}(V \otimes V \otimes V)$ vs 

↓ as endoms

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

in our case: $R''_{12}\, R(\lambda_1)_{13}\, R(\lambda_2)_{23}$

$= R(\lambda_2)_{23}\, R(\lambda_1)_{13}\, R''_{12}$

pictorial pf is given algebraically in 7.5.3 of Chari–Pressley.