Definition: finite ext'n L/K (Henselian w.r.t. valuation v (e.g. complete)) is "unramified" if residue fields λ, κ are such that λ/κ is separable (automatic if finite) and

$$[L:K] = [\lambda:\kappa] \quad \text{i.e.} \quad e=1 \text{ in identity } [L:K] = e\cdot f$$

Proposition: The composite of two unramified extensions is unramified.

Lemma:

$$\begin{tikzcd} & L' = \langle K', L\rangle & \\ K' \arrow[ur, dash] & & L \arrow[ul, dash] \\ & K \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

then L/K unramified $\Rightarrow$ L'/K' unramified.

Tower of extensions in fixed algebraic closure

pf of lemma: λ/κ is finite separable ext'n, so $\lambda = \kappa(\bar\alpha)$ with lift $\alpha \in \mathcal{O}_L$. min. poly. for α: $f(x)/K$; $\bar f(x)$ reduction in $\kappa[x]$

claim that $L = K(\alpha)$: indeed we have

string of inequalities: $[\lambda:\kappa] \leq \deg(\bar f)$

$= \deg(f)$

$= [K(\alpha):K] \leq [L:K] \underset{\uparrow}{=} [\lambda:\kappa]$

unramified assumption

so all ineq. in chain are equalities!

in particular, $L = K(\alpha)$, and $\bar f$ is min. poly for $\bar\alpha/\kappa$

Now $L' = K'(\alpha)$. If g: min poly for α/K'

$\bar g$: reduction in $\kappa'[x]$

then we have $[\lambda':\kappa'] = f \leq [L':K'] = \deg g = \deg \bar g = [\kappa'(\bar\alpha):\kappa']$

$\leq [\lambda':\kappa']$

Key equality: follows b/c $\bar g$ separable as $\bar g \mid \bar f$ as polys in $\kappa'[x]$ and $\bar f$ separable since λ/κ separable ext'n

So if $\bar g = \bar h_1 \cdot \bar h_2$, know $\gcd(\bar h_1, \bar h_2) = 1$

then by Hensel's Lemma g is reducible ↯ minimality of g.

Conclusion: $\bar g$ is irreducible, hence minimal for $\bar\alpha$.

This proves L'/K' is unramified. In particular if

$$\begin{array}{c} L' \\ | \\ K' \\ | \\ K \end{array}$$

with L'/K unramified ($L = L'$ special case) then L'/K' unramified.

But then K'/K must also be unramified ~~...~~ by repeated use of ~~...~~ degree identity $\deg(L/K) = ef$ ~~...~~ with $L : L'$ or K', $K : K'$ or K

*

pf of <u>proposition</u>: If L/K unramified, lemma implies $\langle L, L' \rangle / L'$

$$\begin{tikzcd} & LL' \arrow[dl, dash] \arrow[dr, dash] & \\ L' \arrow[dr, dash] & & L \arrow[dl, dash] \\ & K & \end{tikzcd}$$

But then LL'/K unramified since L'/K unramified and can use degree is mult., separability is transitive.

<u>Definition</u>: An infinite (algebraic) extension is unramified if ~~...~~ it is union of unramified finite extensions.

Form composition of all unramified extensions in fixed alg. closure $\bar{K}$ of K.

"maximal unramified extension"

* In fact, proof of lemma can be used to show following classification of unramified extensions:

L/K unramified $\Longleftrightarrow$ $L = K(x)$ with $x \in \mathcal{O}_K$ s.t. $\bar{f}$: red. of min poly in $k[t]$ is <u>separable</u>.

($\Rightarrow$) def'n of unram. includes separability of residue field. so $\exists$ primitive elt. $\bar{x}$, lift to x.

Consider min poly g, show $\bar{g} = \bar{f}$

($\Leftarrow$) Hensel's Lemma: sep $\Rightarrow$ irreducible.

Proposition: L/K finite extension, then L/K unramified ($e=1$, λ/κ separable)

$\Longleftrightarrow$ $L = K(x)$ for $x \in \mathcal{O}_L$ with $\bar{f}_x \in \kappa_K[t]$ separable.

Pf. L/K unramified $\Rightarrow$ λ/κ_K separable, so have primitive elt. thm.

Find $\bar{x}$ generating λ_L over κ_K. Lift to any $x \in \mathcal{O}_L$.

Let g: min poly. of x, $\in \mathcal{O}_K[t]$ since x integral over K. ($\mathcal{O}_L$ is integral closure of $\mathcal{O}_K$ in L)

and $\deg(g) = [L:K] \underset{\substack{\uparrow \\ \text{unram.} \\ \text{assumption}}}{=} [\lambda_L : \kappa_K]$ so $\bar{g}$ must be minimal poly of $\bar{x}$

(it satisfies $\bar{g}(\bar{x}) = 0$ and $\deg(\bar{g}) = \deg(g)$)

In particular $\bar{g}$ separable as λ_L/κ_K separable.

($\Leftarrow$) If $\bar{f}_x$ separable, then $\bar{f}$ reducible $\Rightarrow$ f reducible by Hensel's Lemma (contradiction!)

($\bar{f}_x$: reduction of min. poly. f_x with coeffs. in $\mathcal{O}_K$)

so $\bar{f}_x$ irreducible and so $\kappa_K(\bar{x})/\kappa_K$ is separable and has degree $= \deg(\bar{f}) = \deg(f) = [L:K]$

so L/K unramified.

Result suggests how to create unique unram. ext'n L/K (in some fixed alg ext'n) $\longleftrightarrow$ finite sep. ext'n of residue field κ_K.

Given $\kappa'/\kappa_K = \kappa_K(\bar{x})/\kappa_K$ with min poly $\bar{g}$. Lift: g, monic coeffs. in κ_K with root x in alg. ext'n and $x \equiv \bar{x} \mod \mathfrak{p}_K$

Consider $L = K[t]/g(t) = K(x)$: unramified by proposition.

It is unique because, if K', K'' are unram. extensions of K both with residue field λ, then the compositum $K' \cdot K''$ also has residue field λ and is unramified by earlier result.

Hence $[K' \cdot K'' : K] = [\lambda : \kappa_K] = [K' : K]$ so $K' = K''$.

Example: K: local field of char. 0 (finite ext'n of $\mathbb{Q}_p$) then residue field κ_K has order $q = p^r$, some r. Finite extensions of $\mathbb{F}_q$ ~~[illegible]~~ of deg n are the splitting fields of $X^{q^n} - X$ over $\mathbb{F}_q$.

(Galois ext'ns with cyclic Galois gp. generated by $x \mapsto x^q$ "Frobenius elt.")

So by the above correspondence, unramified extns of K are likewise splitting fields of $X^{q^n} - X$ with cyclic Galois gp of order n and canonical generator σ "Frobenius"

with $\sigma(\alpha) = \alpha^q \pmod{\mathfrak{P}_n}$ $\forall$ $\alpha \in \mathcal{O}_{K_n}$

where $\mathfrak{P}_n$: maximal ideal in K_n / K

Can go further $\rightarrow$ "tamely ramified extensions"

L/K with λ_L / κ_K separable, T: max. unram. ext'n of K in L (by above $\longleftrightarrow$ κ_s: sep. closure of κ_K in λ_L)

Ask that $[L : T]$ is relatively prime to $p = \mathrm{char}(\kappa_K)$

prove similarly that compositum of tamely ram is tamely ramified.