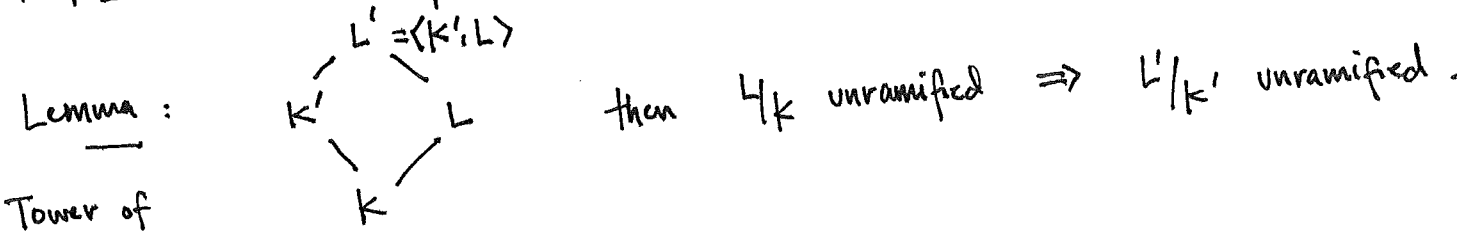


Definition: finite ext'n  $L/K$  is "unramified" if residue fields  $\lambda, k$  are such that  $\lambda/k$  is separable (automatic if finite) and

$[L:K] = [L:\lambda]$  i.e.  $e=1$  in identity  $[L:K] = e \cdot f$

Proposition: The composite of two unramified extensions is unramified.



Tower of extensions in fixed algebraic closure

pf of lemma:  $L/K$  is finite separable ext'n, so

$L = K(\alpha)$

with lift  $\alpha \in \mathcal{O}_L$ .

min. poly. for  $\alpha = f(x)/K$

$\bar{f}(x)$  reduction in  $k[x]$

claim that  $L = K(\alpha)$ : indeed we have

string of inequalities:  $[L:K] \leq \deg(\bar{f}) = \deg(f)$

in particular,  $L = K(\alpha)$ ,

and  $\bar{f}$  is min. poly for  $\bar{\alpha}/k$

$[L:K] = [K(\alpha):K] \leq [L:K] = [L:K]$

unramified assumption

so all ineq. in chain are equalities!

Now  $L' = K'(\alpha)$  if  $g = \text{min poly for } \alpha/K'$   
 $\bar{g} = \text{reduction in } k'[x]$

then we have  $[L':K'] = f \leq [L':K'] = \deg g = \deg \bar{g} = [K'(\bar{\alpha}):K'] \leq [L':K']$

So if  $\bar{g} = \bar{h}_1 \cdot \bar{h}_2$ , know  $\gcd(\bar{h}_1, \bar{h}_2) = 1$

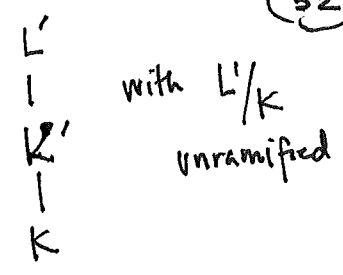
then by Hensel's Lemma

$g$  is reducible  $\nRightarrow$  minimality of  $g$ .

Conclusion:  $\bar{g}$  is irreducible, hence minimal for  $\bar{\alpha}$ .

Key equality: follows b/c  $\bar{g}$  separable as  $\bar{g} | \bar{f}$  as polys in  $k'[x]$  and  $\bar{f}$  separable since  $L/K$  separable ext'n

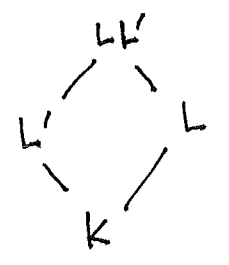
This proves  $L'/K'$  is unramified. In particular if  
 then  $L'/K'$  unramified. ( $L=L'$ ) special case



But then  $K'/K$  must also be unramified ~~also~~ by

repeated use of ~~lemma~~ degree identity  $\deg(L/K) = ef$  with ~~lemma~~  $L: L' \text{ or } K'$   
 $K: K' \text{ or } K$

pf of proposition: If  $L/K$  unramified, lemma implies  $\langle L, L' \rangle / L'$



But then  $LL'/K$  unramified since  $L'/K$  unramified  
 and can use degree is mult., separability is transitive.

Definition: An infinite (algebraic) extension is unramified if ~~it~~ it is union of unramified finite extensions.

Form composition of all unramified extensions in fixed alg. closure  $\bar{K}$  of  $K$ .  
 "maximal unramified extension"

\* In fact, proof of lemma can be used to show following classification of unramified extensions:

$$L/K \text{ unramified} \iff L = K(x) \text{ with } x \in \mathcal{O}_K \text{ s.t. } \bar{f} = \text{red. of min poly in } K[t]$$

( $\implies$ ) def'n of unram. includes separability, so  $\exists$  primitive elt.  $\bar{x}$ , lift to  $x$ . is separable.

Consider min poly  $g$ , show  $\bar{g} = \bar{f}$

( $\impliedby$ ) Hensel's Lemma: sep  $\implies$  irreducible.

Proposition:  $L/K$  finite extension, then  $L/K$  unramified ( $e=1, \lambda/K$  separable)

$\Leftrightarrow L = K(x)$  for  $x \in \mathcal{O}_L$  with  $\bar{f}_x \in K_K[t]$  separable.

pf.  $L/K$  unramified  $\Rightarrow \lambda_L/K_K$  separable, so have primitive elt. thm.

find  $\bar{x}$  generating  $\lambda_L$  over  $K_K$ . Lift to any  $x \in \mathcal{O}_L$ .

Let  $g$ : min poly. of  $x, \in \mathcal{O}_K[t]$  since  $x$  integral over  $K$ . ( $\mathcal{O}_L$  is integral closure of  $\mathcal{O}_K$  in  $L$ )

and  $\deg(g) = [L:K] \stackrel{\text{unram. assumption}}{=} [\lambda_L:K_K]$  so  $\bar{g}$  must be minimal poly of  $\bar{x}$  (it satisfies  $\bar{g}(\bar{x})=0$  and  $\deg(\bar{g}) = \deg(g)$ )

In particular  $\bar{g}$  separable as  $\lambda_L/K_K$  separable.

( $\Leftarrow$ ) If  $\bar{f}_x$  separable, then  $\bar{f}$  reducible  $\Rightarrow f$  reducible by Hensel's Lemma (contradiction!)  
( $\bar{f}_x$ : reduction of min. poly.  $f_x$  with coeffs. in  $\mathcal{O}_K$ )

so  $\bar{f}_x$  irreducible and so  $K_K(\bar{x})/K_K$  has degree =  $\deg(\bar{f}) = \deg(f) = [L:K]$   
is separable and

so  $L/K$  unramified. (in some fixed alg extn)

Result suggests how to create unram. extn  $L/K \leftrightarrow$  finite sep. extn of residue field  $K_K$ .

Given  $K'/K_K = K_K(\bar{x})/K_K$  with min poly  $\bar{g}$ . Lift:  $g$ , monic with root  $x$  in alg. extn and  $x \equiv \bar{x} \pmod{\mathfrak{p}_K}$

Consider  $L = K[t]/g(t) = K(x)$ : unramified by proposition.

It is unique because, if  $K', K''$  are unram. extensions of  $K$  both with residue field  $\lambda$ , then the composition  $K' \cdot K''$  also has residue field  $\lambda$  and is unramified by earlier result.

Hence  $[K' \cdot K'' : K] = [\lambda : k_K] = [K' : K]$  so  $K' = K''$ .

Example:  $K$ : local field of char. 0 (finite ext'n of  $\mathbb{Q}_p$ )

then residue field has order  $q = p^r$ , some  $r$ . Finite extensions of  $k_K$

of  $\mathbb{F}_q$  ~~extension~~ of deg  $n$  are the splitting fields of  $X^{q^n} - X$  over  $\mathbb{F}_q$ .

(Galois ext'ns with cyclic Galois gp. generated by  $x \mapsto x^q$  "Frobenius elt.")

So by the above correspondence, unramified ~~ext'ns~~ ext'ns of  $K$

are likewise splitting fields of  $X^{q^n} - X$  with cyclic Galois

gp of order  $n$  and canonical generator  $\sigma$  "Frobenius"

with  $\sigma(\alpha) = \alpha^q \pmod{\mathfrak{f}_n}$   $\forall \alpha \in \mathcal{O}_{K_n}$

where  $\mathfrak{f}_n$ : maximal ideal in  $K_n/K$

Can go further  $\rightarrow$  "tamely ramified extensions"

$L/K$  with  $\lambda_L/k_K$  separable,  $T$ : max. unram. ext'n of  $K$  in  $L$  (by above  $\leftrightarrow$   $K_S$ : sep. closure of  $K_K$  in  $\lambda_L$ )

Ask that  $[L:T]$  is relatively prime to  $p = \text{char}(k_K)$

prove similarly that composition of tamely ram is tamely ramified.