It is unique because, if $K', K''$ are unram. extensions of $K$ both with residue field $\lambda$, then the compositum $K' \cdot K''$ also has residue field $\lambda$ and is unramified by earlier result.

Hence $[K' \cdot K'' : K] = [\lambda : k_K] = [K' : K]$ so $K' = K''$.

Example: $K$: local field of char. 0 (finite ext'n of $\mathbb{Q}_p$)

then residue field $k_K$ has order $q = p^r$, some $r$. Finite extensions of $\mathbb{F}_q$ ~~____~~ of deg $n$ are the splitting fields of $X^{q^n} - X$ over $\mathbb{F}_q$.

(Galois ext'ns with cyclic Galois gp. generated by $x \mapsto x^q$ "Frobenius elt.")

So by the above correspondence, unramified ext'ns of $K$ are likewise splitting fields of $X^{q^n} - X$ with cyclic Galois gp of order $n$ and canonical generator $\sigma$ "Frobenius"

with $\sigma(\alpha) = \alpha^q \pmod{\mathfrak{P}_n}$ $\forall\ \alpha \in \mathcal{O}_{K_n}$

where $\mathfrak{P}_n$: maximal ideal in $K_n/K$

Can go further $\to$ "tamely ramified extensions"

$L/K$ with $\lambda_L / k_K$ separable, $T$: max. unram. ext'n of $K$ in $L$ (by above $\longleftrightarrow$ $k_s$: sep. closure of $k_K$ in $\lambda_L$)

Ask that $[L:T]$ is relatively prime to $p = \mathrm{char}(k_K)$

prove similarly that compositum of tamely ram is tamely ramified.

Just as with global fields, intermediate extensions:

$$K \subseteq T : \text{max. unram in } L \subseteq V : \text{max. tamely ram. in } L \underbrace{\subseteq L}_{\text{"wild ramification"}}$$

$$\kappa_K \subseteq \lambda_s : \text{max. sep ext'n} = \lambda_s \subseteq \lambda$$

$$v_K(K^\times) = v_T(T^\times) \subseteq v_L(V^\times) \subseteq v_L(L^\times)$$

$$v_L(V^\times) = v_L(L^\times)^{(p)} = \{ \alpha \in v_L(L^\times) \mid m \cdot \alpha \in v_K(K^\times) \}$$

$(m,p)=1$

$m \in \mathbb{N}$.

Return to our earlier example:

$\mathbb{Q}_p(\xi) / \mathbb{Q}_p$ $\xi$ : $n^{th}$ root of unity.

Two extreme cases. $(n,p)=1$, then $\exists$ smallest $f$ s.t. $n \mid p^f - 1$

(i.e. order of $p \pmod n$)

then $\mathbb{Q}_p(\xi)$ is unramified extension with degree $f$ over $\mathbb{Q}_p$.

if $n = p^m$, then $\mathbb{Q}_p(\xi)/\mathbb{Q}_p$ is "totally ramified" — no proper unramified ext'ns of $\mathbb{Q}_p$.

of degree $\varphi(p^m) = p^{m-1} \cdot (p-1)$

Now let $K$ be field with valuation $v$ (or $|\cdot|$ if Archimedean)

and let $K_v$ denote its completion w.r.t. $v$.

$\overline{K_v}$ : alg. closure, $\overline{v}$ — unique extension as $K_v$ : complete, $\overline{K}_v$ : alg. ext'n

$|$

$K_v$ : $v_K$ — canonical extension (as limit of valuations)

$|$

$K$ : $v_K$

This is very reminiscent of early proofs using geometry of numbers, but not using only $\mathbb{R}, \mathbb{C}$ embeddings, but for arbitrary complete local ring.

If $L/K$ algebraic extension, let $\tau : L \to \overline{K_v}$ be a $K$-embedding.

Then we may attach valuation to $L$ by $v_L = \overline{v} \circ \tau$

(and valuation extends $v_K$ since $\tau$ is a $K$-embedding)

For finite extensions $L/K$, then let $L_{v_L}$ : completion of $L$ w.r.t. $v_L$

then $\tau$ extends to continuous map

$$\tau : L_{v_L} \to \overline{K_v}$$

$$x = v_L\text{-}\lim_{n\to\infty} x_n \longmapsto \tau(x) = \overline{v}\text{-}\lim_{n\to\infty} \tau(x_n)$$

if $L/K$ infinite take $L_{v_L}$ to be union of all finite subextensions' completions.

So we have the extensions of fields setting $w := v_L$.

$$\begin{tikzcd} & L_w \arrow[d, dash] \\ L \arrow[ur, dash] \arrow[d, dash] & K_v \\ K \arrow[ur, dash] & \end{tikzcd}$$

and valuation on $L_w$ extends that on $K_v$ so must be unique ext'n in $L_w/K_v$.

In particular if $L_w/K_v$ finite extension, deg. $n$ then

$$|x|_w = \sqrt[n]{|N_{L_w/K_v}(x)|_v}$$

Also $L_w = L \cdot K_v$ since $L/K$ finite

$\Rightarrow$ $L \cdot K_v$ complete and contains $L$, and $\subseteq L_w$.

part of unique ext'n theorem.

Just as we wanted to identify conjugate embeddings in $\mathbb{C}$, want to do the same in $\overline{K}_v$. What are conjugate embeddings? conjugation is just non-trivial elt. of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$

Now take $\sigma \in \mathrm{Gal}(\overline{K}_v/K_v)$.

say $\tau, \tau'$ : two $K$-embeddings of $L/K$ are "conjugate" if $\exists\ \sigma \in \mathrm{Gal}(\overline{K}_v/K_v)$ such that $\tau' = \sigma \circ \tau$.

<u>Classification theorem</u> (version 1) $L/K$ algebraic. $v$: valuation of $K$, Every ext'n of $v$ to $L$ is given by $\bar{v} \circ \tau$, $\tau$: $K$-embedding of $L \to \overline{K}_v$ and two valuations $\bar{v} \circ \tau$, $\bar{v} \circ \tau'$ are equal iff $\tau, \tau'$ are conjugate.

<u>classification theorem</u> (version 2) ~~[illegible]~~ $L = K(\alpha)$ (e.g. $L/K$ separable) with $f(x) \in K[X]$ the minimal poly. of $\alpha$.

with $f(x) = f_1(x)^{m_1} \cdots f_r(x)^{m_r}$ irreducible factors $f_i$ in $K_v[X]$.
(~~[illegible]~~ all ~~[illegible]~~ $m_i = 1$ if $f$ separable)

then valuations of $L$ extending $v$ are in 1-1 correspondence with irreducible factors $f_i$.

<u>pf</u>: (version 1) $\Rightarrow$ (version 2): $K$-embeddings $\tau : L \to \overline{K}_v$ are given by zeros $\beta$ of $f(x)$ lying in $\overline{K}_v$: so $K$-embedding def'd by sending $\alpha \longmapsto \beta$ for each $\beta$. Two embeddings are conjugate precisely when the zeros $\beta, \beta'$ are from the same irreducible factor $f_i$. //

proof of classification 1: 1 Every extension $w = \bar{v} \circ \tau$, $\tau$ : K-embedding:

Assume $L/K$ finite. If $L/K$ infinite, proof is same, but meaning of $L_w$ changes.
$w$ : valuation on $L$, $v$: val. on $K$.

Then any $K_v$-embedding $\tau : L_w \to \bar{K}_v$ (which exists since $L_w = K_v \cdot L$, $L/K$ alg.)

gives valuation $\bar{v} \circ \tau$ extending $v$ on $K_v$, so must equal

$w$ = the unique ext'n of $K_v$-valuation $v$ to $L_w$.

$$\begin{tikzcd} & L_w \arrow[dl, dash] \arrow[d, dash] \\ L \arrow[d, dash] & K_v \\ K \arrow[ur, dash] & \end{tikzcd}$$

If we restrict $w$ to $L$, then $\tau$ restricts to

K-embedding $\tau : L \to \bar{K}_v$ with $w = \bar{v} \circ \tau$.

2 ~~both~~ equal ~~on~~ iff $\tau, \tau'$ conjugate.
$\bar{v} \circ \tau$,
$\bar{v} \circ \tau'$

($\Leftarrow$) Given $\tau$ : embedding, $\sigma \in \mathrm{Gal}(\bar{K}_v / K_v)$, consider whether $\tau$, $\sigma \circ \tau$ are equal.

$\bar{v}$ is unique extension from $K_v$ to $\bar{K}_v$,
of $v$

so for any $\sigma \in \mathrm{Gal}(\bar{K}_v / K)$, $\bar{v} = \bar{v} \circ \sigma$ $\Rightarrow$ $\bar{v} \circ \tau = \bar{v} \circ \sigma \circ \tau$
i.e. valuations are equal.

($\Rightarrow$) Given $\tau, \tau' : L \to \bar{K}_v$ s.t. $\bar{v} \circ \tau = \bar{v} \circ \tau'$

idea: produce K-isomorphism of $\tau L \to \tau' L$, extend to autom. of $\bar{K}_v$, $\sigma$, over K with desired property.

Take $\sigma = \tau' \circ \tau^{-1}$. Then want to extend to

map $\sigma : \tau L \cdot K_v \longrightarrow \tau' L \cdot K_v$

since $\tau L$ is dense in $\tau L \cdot K_v$ as $\tau L$ contains $K$.

Thus every elt $x \in \tau L \cdot K_v$ is expressible as limit of elts in $\tau L$,

i.e. $x = \lim_{n \to \infty} \tau x_n$, $x_n \in L \ \forall n$

Since $\bar{v} \circ \tau = \bar{v} \circ \tau'$ then $\{\tau' x_n\} = \{\sigma \tau x_n\}$ converges in $\bar{v} \circ \tau'$ topology of $\tau' L \cdot K_v$

Call resulting limit $\sigma x$.

Then $x \mapsto \sigma x$ is our desired isomorphism $\tau L \cdot K_v \to \tau' L \cdot K_v$.

and it leaves $K_v$ fixed. Now extend to autom. $\tilde{\sigma}$ of $\mathrm{Gal}(\bar{K}_v / K_v)$.

why is such an extension guaranteed? One alternative is to say equivalent/conjugate extensions require isom. of compositums $\tau L \cdot K_v \to \tau' L \cdot K_v$ in fixed alg. closure.

Then $\tau' L \cdot K_v / K_v$ is finite extn. so has unique abs. value extending one on $K_v$.

(As in Jacobson, B.A. II p. 585)

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Fancy algebraic formulation: tensor products of vector spaces –

Have natural homomorphisms

$$L \otimes_K K_v \longrightarrow L_w = L \cdot K_v$$

$$a \otimes b \longmapsto ab \quad \text{(or maybe better } \tau(a)\, b\text{)}$$

to emphasize K-embedding

L as K-vector space now viewed as $K_v$-vector space (extension of scalars)

If we do this for all places $w$ over $v$ for $L$,

then have map

$$\phi : L \otimes_K K_v \longrightarrow \prod_{w \mid v} L_w$$

Proposition: If $L/K$ separable, then $\phi$ is isomorphism.