Infinite Galois theory / Formalism of class field theory in a way that applies equally well to local / global fields (see that better fit is with local fields, whose structure is much simpler)

Last week, if our Galois ext'n $L/K$ is infinite, give the resulting Galois gp the Krull topology, with basis of open sets of $\sigma \in \mathrm{Gal}(L/K)$ is

$\{ \sigma \cdot \mathrm{Gal}(L/M) \}_{M:\ \text{finite, Galois}/K \text{ subext. of } L}$. Stated but didn't prove that $\mathrm{Gal}(L/K) =: G$ is compact, Hausdorff with 1-1 corresp. between subfields and closed subgps of $G$.

Compactness:
$$\pi : G \longrightarrow \prod_{M:\ \text{finite, Galois}/K} \mathrm{Gal}(M/K) \quad \leftarrow \text{compact upon giving discrete topology to all finite gps.}$$
$$\sigma \longmapsto \prod_M \sigma|_M$$

show $\pi$ continuous, injective, with $\pi(G)$ closed.

Note in Krull topology, taking $M$: Galois (i.e. $M$ normal) so $\mathrm{Gal}(L/M)$ is a normal subgp. of $\mathrm{Gal}(L/K)$.

Motivated by this, define "profinite gp" to be a topological gp. which is Hausdorff, compact, and has base of nbhds of 1 that are <u>normal</u> subgps.

($\Longleftrightarrow$ $G$ totally disconnected (i.e. conn. comp. of any pt. is itself.))

Typical Neukirch definition - uses properties desired rather than explicit construction.

But as usual, can realize it by algebraic construction: projective limit. (seen before in context of $\mathbb{Z}_p$, etc.)

Prop. N.2.8 : $G$ profinite, then $G \simeq \varprojlim_N G/N$, $N$: open normal subgps.

and conversely $\varprojlim_i G_i =: G$ is profinite for any projective system $\{G_i\}$

so in our example of profinite $Gal(L/K)$, then the proposition gives

$$Gal(L/K) \simeq \varprojlim_{\substack{M:\text{Galois}\\ \text{finite}}} Gal(M/K) \qquad \text{since } Gal(M/K) = Gal(L/K)/Gal(L/M)$$

Simplest concrete <u>example</u>: $K = \mathbb{F}_q$. $\mathbb{F}_{q^n}/\mathbb{F}_q$ give projective system with Galois gps $\simeq \mathbb{Z}/n\mathbb{Z}$

Frob: $z \mapsto z^q \longmapsto 1 \pmod{n}$
generators map to gens.

$$\text{thus } Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \varprojlim_n Gal(\mathbb{F}_{q^n}/\mathbb{F}_q)$$
"absolute Galois gp"

$$= \varprojlim_n \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}$$

Neukirch does 9 examples in Section 2. <u>Plan</u>: Develop formalism of CFT through abstract profinite gps.

Always want to keep main example of inf. Galois gp in mind. So index closed subgroups by set called "fields" $G_K$ : closed in $G$, $K$ = field. ("fixed field" of $G_K$) with $k$ s.t. $G_k = G$ the "base field", $\bar{k}$ s.t. $G_{\bar{k}} = \{e\}$, write $K \subseteq L$ for "fields" formally if $G_L \subseteq G_K$, write $L/K$ "finite" if $G_L$ of finite index in $G_K$, i.e. open and index is formal degree.

---

Study $G$-modules of form: $A$: mult. abelian gp (e.g. mult gp. of field)

$G$ acts like Galois automs.: $\sigma \in G$: $a \mapsto a^\sigma$

Since $G$ has topology, want action to be continuous: $G \times A \to A$, $(\sigma, a) \mapsto a^\sigma$ is continuous map when $A$ is given discrete topology.

Find, for any $(\sigma, a)$, an open subgp. $\mathcal{U} = G_K$ of $G$ such that open set $\sigma \cdot \mathcal{U} \times \{a\}$ of $(\sigma, a)$ is mapped to the open set $\{a^\sigma\}$, i.e. $a^\sigma \in A^{\mathcal{U}}$ elts. fixed by $\mathcal{U}$.

Since $A^{u} = A^{G_K}$ with $K/k$ finite then we can guarantee this if we assume (i.e. require) $A = \bigcup_{[K:k]<\infty} A^{G_K}$. Then any open set in $A$ consists of union of pts, each in some $A^{G_K}$ with inverse image open.

—

If $L/K$ extension of fields $A^{G_K} \subseteq A^{G_L}$. If, in particular, $L/K$ finite, then there is a norm map: $N_{L/K}: A^{G_L} \longrightarrow A^{G_K}$ with

$$a \longmapsto \prod_{\sigma} a^{\sigma}$$

$\sigma$ varying over reps of $G_L \backslash G_K$.

If $L/K$ Galois, then $A^{G_L}$ is a $Gal(L/K)$-module, with $(A^{G_L})^{Gal(L/K)} = A^{G_K}$

Two key groups in formal class field theory: ① $A^{G_K} / N_{L/K}(A^{G_L})$ "norm residue gp"

$$=: H^0(Gal(L/K), A^{G_L})$$

② $A^{G_L}_{(1)} / I_{Gal(L/K)} A^{G_L}_{(1)} =: H^{-1}(Gal(L/K), A^{G_L})$

where $A^{G_L}_{(1)} = \{ a \in A^{G_L} \mid N_{L/K}(a) = 1 \}$ "norm-one gp."

$$I_{Gal(L/K)} A^{G_L}_{(1)} = \langle a^{\sigma} \cdot a^{-1} \mid a \in A^{G_L} \rangle$$

Assume $G, A$ chosen such that the following is satisfied:

Axiom: $H^{-1}(Gal(L/K), A^{G_L}) = 1$ for all finite extensions $L/K$.

Then we establish several 1-1 correspondences, ingredients in later statements of class field theory.

Final ingredient is surjective $G$-homom. $\wp : A \to A$
module
$a \mapsto a^{\wp}$

with cyclic kernel $\mu_{\wp}$.

Notation meant to suggest most important special case: $n^{th}$ power map $a \mapsto a^n$

where $\mu_{\wp} = \mu_n$, $n$-th rts. of unity.

In general, $\# \mu_{\wp}$ is called the "exponent" of $\wp$.

Use this to define Kummer extensions w.r.t. $\wp$:

Fix $K$ s.t. $\mu_{\wp} \subseteq A^{G_K}$. For every $B \subseteq A$, let

$K(B)$ be the fixed field of closed subgp. $H := \{ \sigma \in G_K \mid b^{\sigma} = b \ \forall b \in B \}$

In particular $K(B)$ is Galois ext'n $/K$ if $B$ is $G_K$-invariant.

Then Kummer ext'n is just special case $B = \wp^{-1}(\Delta)$ for some subset $\Delta \subseteq A^{G_K}$

Defines abelian Galois ext'n of exponent $n$; as

$$\mathrm{Gal}(K(\wp^{-1}(a))/K) \longrightarrow \mu_{\wp} \quad \text{is injective}$$

$$\sigma \longmapsto \alpha^{\sigma - 1} \quad \text{where } \alpha \in \wp^{-1}(a)$$

so $\mathrm{Gal}(K(\wp^{-1}(\Delta))/K) \longrightarrow \prod_{a \in \Delta} \mathrm{Gal}(K(\wp^{-1}(a))/K) \longrightarrow \underbrace{\mu_{\wp}^{\Delta}}_{\Delta \text{ many copies}}$.

is injective homom into abelian gp.

Converse is also true : if $L|K$ an abelian extension with exponent $n$ (so $\sigma^n = 1 \ \forall \ \sigma \in Gal(L/K)$)

then $L = K(\wp^{-1}(\Delta))$ with

$$\Delta = \underbrace{A_L^{\wp}}_{(A^{G_L})^{\wp}} \cap \underbrace{A_K}_{A^{G_K}} \quad \text{for some homom. } \wp \text{ with "exponent" } n.$$

If $L/K$ cyclic then $L = K(\alpha)$ with $\alpha^{\wp} = a \in A^{G_K}$.

Main <u>Thm. of Kummer Thy</u>: The map $\Delta \longmapsto L = K(\wp^{-1}(\Delta))$

gives 1-1 corresp. between groups $\Delta$ s.t. $(A^{G_K})^{\wp} \leq \Delta \leq A^{G_K}$

and abelian ext'ns of exponent $n$.

If $\Delta \longleftrightarrow L$ then $A_L^{\wp} \cap A_K = \Delta$ and $\exists$ canonical isom.

$$\Delta / A_K^{\wp} \simeq Hom(Gal(L/K), \mu_{\wp})$$

$$a \bmod A_K^{\wp} \longmapsto \left[ \chi_a : \sigma \mapsto \alpha^{\sigma - 1} \right]$$

with $\alpha \in \wp^{-1}(a)$.

<u>Primary example</u>: $G := Gal(\bar{k}/k)$, $A = \bar{k}^{\times}$ mult. gp. of alg. closure.

$\wp : a \mapsto a^n$ $\quad$ $gcd(n, char(k)) = 1$ (arb. if $char(k) = 0$)

then our axiom : $L/K$ finite extension, then

$$H^{-1}(Gal(L/K), \underbrace{(\bar{K}^*)^{G_L}}_{L^{\times}}) = 1$$

is famous theorem "Hilbert 90".

<u>Corollary</u>: $n \in \mathbb{N}$, $\gcd(n, \mathrm{char}(K)) = 1$. Suppose $\mu_n \subseteq K$.

Then abelian extns of exponent $n$ $\overset{1-1}{\longleftrightarrow}$ $\Delta \subseteq K^\times$ with $(K^\times)^n \subseteq \Delta$
$L/K$

via the map $\Delta \longmapsto L = K(\underbrace{\sqrt[n]{\Delta}}_{\text{adjoin } n^{\text{th}} \text{ roots of elts of } \Delta.})$ and $\mathrm{Gal}(L/K) \simeq \mathrm{Hom}(\Delta/(K^\times)^n, \mu_n)$