Continuing abstract development of CFT via profinite gps:

G : profinite. Now let d : surjection (and cont. homom.) of profinite gps from $G \longrightarrow \hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z}$

This is our abstraction of valuation.

If $G = \mathrm{Gal}(\bar{k}/k)$ $\quad \bar{k}$: sep. closure of non-arch. local field k

then surjection $d : G \twoheadrightarrow \mathrm{Gal}(\tilde{k}/k) \simeq \hat{\mathbb{Z}}$ where $\tilde{k}$: max. unram. extension.

Remember that we had defined Frobenius autom. φ in $\mathrm{Gal}(\tilde{k}/k)$ according to

(remembering finite unram. ext'ns were in 1-1 corresp. with ext'ns of residue field)

For all n : $\quad \alpha^{\varphi} \equiv \alpha^{q} \pmod{\mathfrak{p}_n} \quad \forall\, \alpha \in \mathcal{O}_{K_n}$

where $\mathfrak{p}_n$ is maximal ideal in val. ring $\mathcal{O}_{K_n}$ of K_n/k : unram. of deg. n

Equivalently $\quad \alpha^{\varphi} \equiv \alpha^{q} \pmod{\tilde{\mathfrak{p}}} \quad \forall\, \alpha \in \tilde{\mathcal{O}}$ for $\tilde{k}/k$.

If k arbitrary field (i.e. just formal index for "base field" in profinite gp) then we can set $\tilde{k}$ to be "fixed field" of the kernel I of $d : G \to \hat{\mathbb{Z}}$ so that $\mathrm{Gal}(\tilde{k}/k) = G_k / G_{\tilde{k}} \simeq \hat{\mathbb{Z}}$

If K/k subext'n, then restrict $d : G_K \to \hat{\mathbb{Z}}$ with kernel I_K : "inertia gp" so $I_K = G_K \cap I = G_K \cap G_{\tilde{k}}$ $= G_{K \cdot \tilde{k}}$

so maximal unram. ext'n of K is $K \cdot \tilde{k} =: \tilde{K}$.

Set $f_K := [\hat{\mathbb{Z}} : d(G_K)]$, $e_K := [I : I_K]$

so that $\frac{1}{f_K} \cdot d : G_K \to \hat{\mathbb{Z}}$ is surjective homom.
(provided $f_K < \infty$)

with kernel I_K so that $\frac{1}{f_K} \cdot d : \mathrm{Gal}(\tilde{K}/K) \xrightarrow{\sim} \hat{\mathbb{Z}}$

then frobenius elt. $\varphi_K \in \mathrm{Gal}(\tilde{K}/K)$ is the elt. s.t.

$$\frac{1}{f_K} d(\varphi_K) = 1 \text{ in } \hat{\mathbb{Z}}.$$

Similarly, if L/K ext'n, set inertia degree, ram. index to be

$$f_{L/K} = [d(G_K) : d(G_L)] \qquad e_{L/K} = [I_K : I_L].$$

which is well-behaved in towers and

Proposition <u>IV.4.2</u> : $[L:K] = f_{L/K} \cdot e_{L/K}$.

<u>pf</u>: $[L:K] = |\mathrm{Gal}(L/K)| = [G_K : G_L]$

and we have the commutative diagram of Galois ext'ns

$$\begin{tikzcd}
1 \arrow[r] & I_L \arrow[r] \arrow[d] & G_L \arrow[r] \arrow[d] & d(G_L) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I_K \arrow[r] & G_K \arrow[r] & d(G_K) \arrow[r] & 1
\end{tikzcd}$$

$$\Rightarrow 1 \to I_K/I_L \to \mathrm{Gal}(L/K) \to d(G_K)/d(G_L) \to 1.$$

Even if L/K not Galois, pass to Galois closure M/K, use that e, f well-behaved in towers.

so starting with single homom $d: G \to \hat{\mathbb{Z}}$, we get a whole tower of homoms for each intermediate field and a way of defining inertia, ramification, and $\underset{\sim}{\text{Frobenius}}$ for arbitrary fields

$\uparrow$

so far only have this for unramified extensions.

Even have Frobenius for relative unramified extensions $\begin{matrix} \tilde{K} \\ | \\ L \\ | \\ K \end{matrix}$,

setting $\varphi_{L/K}$ to be image of φ_K under usual surjection $\mathrm{Gal}(\tilde{K}/K) \to \mathrm{Gal}(L/K)$ (assume $f_K < \infty$)

Remarkable fact: Unramified extensions' Frobenii are enough to understand $\mathrm{Gal}(L/K)$ (with $f_K < \infty$).

Idea - given L/K consider $\mathrm{Gal}(\tilde{L}/K)$ where $\tilde{L}$: max. unram $/L$

$$= \tilde{L} \cdot \tilde{K} = L \cdot K \tilde{k} = L \cdot \tilde{k} = \tilde{L}$$

then $\exists$ semigp. $\mathrm{Frob}(\tilde{L}/K)$ defined as set $\{ \sigma \in \mathrm{Gal}(\tilde{L}/K) \mid \underbrace{d_K}_{\frac{1}{f_K} \cdot d(\sigma)}(\sigma) \in \mathbb{N} \}$

(well-defined since $d_K : G_K \to \hat{\mathbb{Z}}$ and factors through $\mathrm{Gal}(\tilde{L}/K)$ since $G_{\tilde{L}} = I_L \subseteq I_K$.)

Two key points: ① L/K finite, Galois then $\mathrm{Frob}(\tilde{L}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$

$$\sigma \longmapsto \sigma|_L$$

② If Σ denotes fixed field of lift $\tilde{\sigma}$ of $\underset{\in \mathrm{Gal}(L/K)}{\sigma}$ under this map, then $\tilde{\sigma} = \varphi_\Sigma$: Frob on Σ

picture :

$$\begin{tikzcd}[arrows=dash]
& \tilde{L} = \tilde{\Sigma} \arrow[dl] \arrow[d] & \\
L \arrow[dr] & \Sigma \arrow[d] & \tilde{K} \arrow[dl] \\
& k &
\end{tikzcd}$$

To such a $G \xrightarrow{d} \hat{\mathbb{Z}}$, and multiplicative abelian gp, (continuous) G-module $A := A_k$

we define "Henselian valuation" of A_k to be

homom. $v : A_k \to \hat{\mathbb{Z}}$ s.t. the following properties are satisfied:

(i) $v(A_k) = Z \supseteq \mathbb{Z}$ and $Z/nZ \simeq \mathbb{Z}/n\mathbb{Z} \quad \forall n \in \mathbb{N}$.

(ii) $v(N_{K/k} A_K) = f_K \cdot \mathbb{Z} \quad \forall$ finite extensions K/k

$A_k^{G_K}$

Remember, $f_K = [\hat{\mathbb{Z}} : d(G_K)]$ so this is dependence on d.

Then we may associate v_K to every K by $v_K := \frac{1}{f_K} v \circ N_{K/k}$

Proof of surjectivity in 1 :

use commutativity of diagram

$$\begin{tikzcd}
G_L \arrow[r, "d_L"] \arrow[d] & \hat{\mathbb{Z}} \arrow[d, "f_{L/K}"] \\
G_K \arrow[r, "d_K"] & \hat{\mathbb{Z}}
\end{tikzcd}$$

to show relation of Frobenii:

$\varphi_L \big|_{\tilde{K}} = \varphi_K^{f_{L/K}}$ this $f_{L/K}$ introduces exponents $n \in \mathbb{N}$.