

Continuing abstract development of CFT via profinite gps:

$G$ : profinite. Now let  $d$ : surjection (and cont. homom.)  
of profinite gps

This is our abstraction of valuation. from  $G \rightarrow \hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z}$

If  $G = \text{Gal}(\bar{k}/k)$   $\bar{k}$ : sep. closure of non-arch. local field  $k$

then surjection  $d: G \rightarrow \text{Gal}(\tilde{k}/k) \cong \hat{\mathbb{Z}}$  where  $\tilde{k}$ : max. unram. extension.

Remember that we had defined Frobenius autom.  $\varphi$

in  $\text{Gal}(\tilde{k}/k)$  according to

For all  $n$ :  $\alpha^\varphi \equiv \alpha^q \pmod{\mathfrak{f}_n} \quad \forall \alpha \in \mathcal{O}_{K_n}$

where  $\mathfrak{f}_n$  is maximal ideal

in val. ring  $\mathcal{O}_{K_n}$  of  $K_n/k$ : unram. of deg.  $n$

(remembering finite unram. extns were in 1-1 corresp. with extns of residue field)

Equivalently  $\alpha^\varphi \equiv \alpha^q \pmod{\tilde{\mathfrak{f}}_n} \quad \forall \alpha \in \tilde{\mathcal{O}}$   
for  $\tilde{k}/k$ .

If  $k$  arbitrary field (i.e. just formal index for "base field" in profinite gp)

then we can set  $\tilde{k}$  to be "fixed field" of the kernel  $I$

of  $d: G \rightarrow \hat{\mathbb{Z}}$  so that  $\text{Gal}(\tilde{k}/k) = G_k/G_{\tilde{k}} \cong \hat{\mathbb{Z}}$

If  $K/k$  subext'n, then restrict  $d: G_K \rightarrow \hat{\mathbb{Z}}$

with kernel  $I_K$ : "inertia gp" so  $I_K = G_K \cap I = G_K \cap G_{\tilde{k}}$

so maximal unram. ext'n of  $K$  is  $K \cdot \tilde{k} =: \tilde{K} = G_K \cdot \tilde{k}$

$$\text{Set } f_K := [\hat{\mathbb{Z}} : d(G_K)], \quad e_K := [I : I_K]$$

so that  $\frac{1}{f_K} \cdot d : G_K \rightarrow \hat{\mathbb{Z}}$  is surjective homom. (provided  $f_K < \infty$ )

with kernel  $I_K$  so that  $\frac{1}{f_K} \cdot d : \text{Gal}(\tilde{K}/K) \xrightarrow{\sim} \hat{\mathbb{Z}}$

then Frobenius elt.  $\varphi_K \in \text{Gal}(\tilde{K}/K)$  is the elt. s.t.

$$\frac{1}{f_K} d(\varphi_K) = 1 \text{ in } \hat{\mathbb{Z}}.$$

Similarly, if  $L/K$  ext'n, set inertia degree, ram. index to be

$$f_{L/K} = [d(G_K) : d(G_L)] \quad e_{L/K} = [I_K : I_L].$$

which is well-behaved in towers and

Proposition IV.4.2 :  $[L:K] = f_{L/K} e_{L/K}$

pf:  $[L:K] = |\text{Gal}(L/K)| = \cancel{[G_L : G_K]} [G_K : G_L]$

and we have the commutative diagram of Galois ext'ns

$$\begin{array}{ccccccc} 1 & \rightarrow & I_L & \rightarrow & G_L & \rightarrow & d(G_L) \rightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & I_K & \rightarrow & G_K & \rightarrow & d(G_K) \rightarrow 1 \end{array} \Rightarrow 1 \rightarrow I_K/I_L \rightarrow \text{Gal}(L/K) \rightarrow d(G_K)/d(G_L) \rightarrow 1.$$

Even if  $L/K$  not Galois, pass to Galois closure  $M/K$ , use that  $e, f$  well-behaved in towers.

so starting with single homom  $\phi: G \rightarrow \hat{\mathbb{Z}}$ , we get a whole tower of homoms for each intermediate field and a way of defining inertia, ramification, and Frobenius for arbitrary fields

↑  
so far only have this for unramified extensions.

Even have Frobenius for relative unramified extensions

$$\begin{array}{c} \tilde{K} \\ | \\ L \\ | \\ K \end{array},$$

setting  $\varphi_{L/K}$  to be image of  $\varphi_K$  under

usual surjection  $\text{Gal}(\tilde{K}/K) \rightarrow \text{Gal}(L/K)$  (assume  $f_K < \infty$ )

Remarkable fact: Unramified extensions' Frobenii are enough to understand  $\text{Gal}(L/K)$  (with  $f_K < \infty$ ).

Idea - given  $L/K$  consider  $\text{Gal}(\tilde{L}/K)$  where  $\tilde{L} = \text{max. unram. } L$

then  $\exists$  semigrp.  $\text{Frob}(\tilde{L}/K)$  defined as

set  $\{ \sigma \in \text{Gal}(\tilde{L}/K) \mid \underbrace{d_K(\sigma)} \in \mathbb{N} \}$

$\perp \cdot d(\sigma)$   
 $f_K$

factors through

$\text{Gal}(\tilde{L}/K)$  since

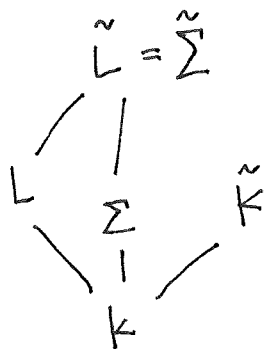
(well-defined since  $d_K: G_K \rightarrow \hat{\mathbb{Z}}$  and

$$G_{\tilde{L}} = I_{\tilde{L}} \subseteq I_K.)$$

Two key points: ①  $L/K$  finite, Galois then  $\text{Frob}(\tilde{L}/K) \twoheadrightarrow \text{Gal}(L/K)$   
 $\sigma \longmapsto \sigma|_L$

② If  $\Sigma$  denotes fixed field of lift  $\tilde{\sigma}$  of  $\sigma$  under this map, then  $\tilde{\sigma} = \varphi_{\Sigma} : \text{Frob on } \Sigma$   
 $\in \text{Gal}(L/K)$

picture:



to such a  $G \xrightarrow{d} \hat{\mathbb{Z}}$ , and multiplicative abelian,  $G$ -module  $A := A_k$  (continuous) gp

we define "Henselian valuation" of  $A_k$  to be

homom.  $v: A_k \rightarrow \hat{\mathbb{Z}}$  s.t. the following properties are satisfied:

(i)  $v(A_k) = \mathbb{Z} \cong \mathbb{Z}$  and  $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \quad \forall n \in \mathbb{N}$ .

(ii)  $v(N_{K/k} A_k) = f_k \cdot \mathbb{Z}$  for finite extensions  $K/k$

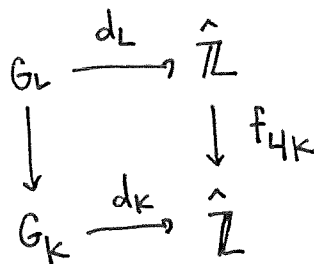
$$\begin{array}{c}
 \uparrow \\
 A_k^{G_k}
 \end{array}$$

Remember,  $f_k = [\hat{\mathbb{Z}} : d(G_k)]$  so this is dependence on  $d$ .

then we may associate  $v_k$  to every  $K$  by  $v_k := \frac{1}{f_k} v \circ N_{K/k}$

Proof of surjectivity in ①:

use commutativity of diagram



to show relation of Frobenius:

$$\varphi_L \Big|_{\tilde{K}} = \varphi_k^{f_{L/k}}$$

this  $f_{L/k}$  introduces exponents  $n \in \mathbb{N}$ .