Abstract theory: G – profinite $\quad d$: surj. homom $G \twoheadrightarrow \hat{\mathbb{Z}}$

a continuous G-module A, and homom $v: A \to \hat{\mathbb{Z}}$ compatible with d.

Call v a "Henselian valuation" because it satisfies, by assumption,

(i) $\quad v(A) = Z \supseteq \mathbb{Z}$ s.t. $Z/nZ \simeq \mathbb{Z}/n\mathbb{Z} \quad \forall\, n \in \mathbb{N}$

(ii) $\quad v(N_{K/k} A^{G_K}) = f_K \cdot \mathbb{Z} \quad \forall$ finite extns K/k.

so if $G = \mathrm{Gal}(\bar{k}/k)$, $\bar{k}$: sep. closure of k. $\quad A = \bar{k}^\times$ so that $A^{G_K} = K^\times$

then (ii) agrees with valuation's unique extension to K/k

$$v_K : A^{G_K} \to Z$$
$$a \mapsto \frac{1}{f_K} v(N_{K/k}(a))$$

provided K/k unramified (so that $[K:k] = f_K$)

define primes of K, units of K as elts with valuation $v_K = 1$ or $= 0$ respectively.

Main Thm. of Class Field Theory: $\exists$ canonical isomorphism

$$r_{L/K} : \mathrm{Gal}(L/K) \xrightarrow{\sim} A^{G_K} / N_{L/K} A^{G_L}$$

when L/K finite abelian extn.

On Wednesday: $\exists$ surjective homom, for L/K with $f_K < \infty$, Galois, finite (not nec. abelian)

$$\eta : \mathrm{Frob}(\tilde{L}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$$
$$\sigma \mapsto \sigma|_L$$

where $\mathrm{Frob}(\tilde{L}/K) := \{ \sigma \in \mathrm{Gal}(\tilde{L}/K) \mid d_K(\sigma) \in \mathbb{N} \}$

"generalized Frobenius elts"

since Frobenius were inverse image of 1 under $d_K : G_K \twoheadrightarrow \hat{\mathbb{Z}}$

(Note: Frobenius is unique mod kernel I_K and $I_K \backslash G_K \simeq \mathrm{Gal}(\tilde{K}/K)$)

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So modest first step: construct canonical homom.

$$r_{\tilde{L}/K} : \mathrm{Frob}(\tilde{L}/K) \longrightarrow A^{G_K} / N_{\tilde{L}/K} A^{G_{\tilde{L}}}$$

for any finite Galois ext'n.

$$\sigma \longmapsto \underset{\sim}{?}$$

map into A^{G_K}
then take induced map to quotient

How to get elt. of A^{G_K} ?
Want to think of this as $K^\times$ in Local CFT.
Take norm down to K. From where? From $\tilde{L}$ obviously too much.

Let Σ : fixed field gen. by σ (for subgp.), π_Σ : prime elt. of $A_\Sigma := A^{G_\Sigma}$ with respect to v_Σ

$$\sigma \longmapsto N_{\Sigma/K}(\pi_\Sigma) \mod N_{\tilde{L}/K} A^{G_{\tilde{L}}}$$

(units will map to $N_{\tilde{L}/K} A^{G_{\tilde{L}}}$ so definition is indep. of choice of prime elt π_Σ)

Show this is a semi-group homom.
(i.e. it is multiplicative. Semi-groups have mult. but not nec. identity elt.)

but only when we have certain vanishing of cohomology gps.
$H^i(\mathrm{Gal}(L/K), \underset{\text{units}}{\underbrace{\mathcal{U}_L}})$ $i=0,-1$

Granting this, which takes several pages to prove,
we can use surjectivity $\mathrm{Frob}(\tilde{L}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$
and containment $N_{\tilde{L}/K} A^{G_{\tilde{L}}} \subseteq N_{L/K} A^{G_L} \ldots$

to construct homom. $\quad r_{L/K} : \mathrm{Gal}(L/K) \longrightarrow A_K / N_{L/K} \cdot A_L$

$$\sigma \longmapsto N_{\Sigma/K}(\pi_\Sigma) \bmod N_{L/K} A_L$$

where Σ is the fixed field of the lift $\tilde{\sigma}$ in $\mathrm{Frob}(\tilde{L}/K)$ of σ in $\mathrm{Gal}(L/K)$

(just check indep. of lift $\tilde{\sigma}$ which is straightforward.)

<u>Remark</u> : $N_{L/K}(a) = \prod\limits_{\substack{\sigma :\text{ reps} \\ \text{for } G_L \backslash G_K}} a^\sigma$. But if extension is infinite then set

$$N_{L/K} A^{G_L} = \bigcap_{\substack{M:\text{finite} \\ \text{subext.}}} N_{M/K} A^{G_M}$$

thus containment $N_{\tilde{L}/K} A^{G_{\tilde{L}}} \subseteq N_{L/K} A^{G_L}$ is by definition.

If in particular L/K is unramified extension, i.e. $e_{L/K} = [\underset{\sim}{I_K} : \underset{\sim}{I_L}] = 1$

kernels of d_K, d_L from $G_K, G_L \to \hat{\mathbb{Z}}$

then $\quad r_{L/K} : \mathrm{Gal}(L/K) \longrightarrow A_K := A^{G_K}$

is determined by action on Frobenius $\varphi_{L/K}$.

First, lift $\varphi_{L/K}$ to $\mathrm{Frob}(\tilde{L}/K)$. Note $\tilde{L}$: max. unram. ext'n of L $= L \cdot \tilde{K} = \tilde{K}$ since assuming

$$\begin{array}{c} \tilde{K} \\ | \\ L \\ | \\ K \end{array}$$

So lift to $\mathrm{Frob}(\tilde{K}/K)$,

and φ_K : Frobenius in $\tilde{K}/K$ is such lift.

What is fixed field Σ for φ_K? Just K.

so $\quad r_{L/K}(\varphi_{L/K}) = \pi_K$

Proposition: L/K unramified then

$$r_{L/K} : \mathrm{Gal}(L/K) \longrightarrow A^{G_K} / N_{L/K} A^{G_L} \quad \text{is an isomorphism.}$$

$$\varphi_{L/K} \longmapsto \pi_K \mod N_{L/K} \cdot A^{G_L}$$

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pf: valuation $v_K : A^{G_K} \to \mathbb{Z}$ takes $N_{L/K} A^{G_L}$ into $n \cdot \mathbb{Z}$

if $[L:K] = n = f_{L/K}$ as the following diagram commutes:

$$\begin{array}{ccc} A^{G_L} & \xrightarrow{v_L} & \mathbb{Z} \subseteq \hat{\mathbb{Z}} \\ \downarrow N_{L/K} & & \downarrow f_{L/K} \\ A^{G_K} & \xrightarrow[v_K]{} & \mathbb{Z} \subseteq \hat{\mathbb{Z}} \end{array}$$

Indeed moving rightward and down, $a \mapsto f_{L/K} \cdot v_L(a)$

But $f_{L/K} \cdot v_L(a) \underset{\text{def}}{=} f_{L/K} \cdot \frac{1}{f_L} v(N_{L/k}(a))$

$$= \frac{1}{f_K} v \cdot (N_{K/k} N_{L/K}(a))$$

$$= v_K(N_{L/K}(a)) \checkmark.$$

So v_K induces map from

$$A^{G_K} / N_{L/K} A^{G_L} \longrightarrow \mathbb{Z}/n\mathbb{Z} \underset{\substack{\uparrow \\ \text{assumption on Henselian valuations}}}{\simeq} \hat{\mathbb{Z}}/n\hat{\mathbb{Z}}$$

$$\pi_K \longmapsto v_K(\pi_K) = 1.$$

so composition $\mathrm{Gal}(L/K) \longrightarrow A^{G_K} / N_{L/K} A^{G_L} \longrightarrow \hat{\mathbb{Z}}/n\hat{\mathbb{Z}}$ is surjective.

$$\varphi_{L/K} \longmapsto \pi_K \longmapsto 1$$

If $v_K(a) \equiv 0 \mod n\mathbb{Z}$ so $a = u \cdot \pi_K^{d \cdot n}$ for some unit u, integer d

By vanishing axiom on cohomology, $u = N_{L/K}(\varepsilon)$ for some $\varepsilon \in U_L$,

so $a = N_{L/K}(\varepsilon \pi_K^d)$ i.e. $a \in N_{L/K} A^{G_L}$. //