<u>Main Theorem</u> of Class Field Theory: For L/K finite, Galois

$$r_{L/K} : \mathrm{Gal}(L/K)^{ab} \longrightarrow A_K / N_{L/K} A_L$$

where $G^{ab} = G/[G,G]$

Commutator subgp. $\{aba^{-1}b^{-1} \mid a,b \in G\}$

is an isomorphism, provided that

$$|H^i(\mathrm{Gal}(L/K), A_L)| = \begin{cases} [L:K] & \text{for } i=0, \\ 1 & \text{for } i=-1. \end{cases}$$

some quotient of elts in A_L of norm 1.

<u>Corollary</u>: $L \mapsto N_{L/K} A_L$ gives 1-1 correspondence between finite abelian ext'ns L and open subgps of A_K. (topology: a has base of nbhds $a \cdot N_{L/K} A_L$, L: finite Galois) with good functorial properties. E.g.,

$$L_1 \subseteq L_2 \iff N_{L_2/K} A_{L_2} \supseteq N_{L_1/K} A_{L_1}.$$

Local class field theory: $A_K = K^\times$ so $A_K / N_{L/K} A_L = K^\times / N_{L/K} L^\times$

Given $\sigma \in \mathrm{Gal}(L/K) \mapsto \tilde{\sigma} \in \mathrm{Gal}(\tilde{L}/K)$ s.t. $d_K(\tilde{\sigma}) \in \mathbb{N}$ i.e. $\tilde{\sigma}|_{\tilde{K}} = \varphi_K^n$, some $n \in \mathbb{N}$

$$\mapsto N_{\Sigma/K}(\pi_\Sigma) \text{ where } \Sigma = \text{fixed field of } \tilde{\sigma}.$$

<u>LCFT Corollary 1</u>: Every finite abelian ext'n $L/\mathbb{Q}_p$ is contained in $\mathbb{Q}_p(\zeta)$, ζ: rt. of unity

<u>LCFT Corollary 2</u>: Every finite abelian ext'n $L/\mathbb{Q}$ is contained in $\mathbb{Q}(\zeta)$, ζ: rt. of unity

(Kronecker-Weber Thm)

Can check that "norm topology" in abstract CFT $\longleftrightarrow$ usual valuation theory topology for local field.

<u>proof of Corollary 1</u>: Find f, n such that $(p^f) \times \mathcal{U}_{\mathbb{Q}_p}^{(n)} \subseteq N_{L/\mathbb{Q}_p} L^\times$

By functoriality of CFT, the "class field" $M \longleftrightarrow$ open sbgp $(p^f) \times \mathcal{U}_{\mathbb{Q}_p}^{(n)}$ contains L

But $(p^f) \times \mathcal{U}_{\mathbb{Q}_p}^{(n)} = \left((p^f) \times \mathcal{U}_{\mathbb{Q}_p} \right) \cap \left((p) \times \mathcal{U}_{\mathbb{Q}_p}^{(n)} \right)$

so it suffices to show M is contained in cyclotomic extension.

so M is the composition of fields corresponding to each of the subgroups.

Ⓐ $(p) \times \mathcal{U}_{\mathbb{Q}_p}^{(n)} \longleftrightarrow \mathbb{Q}_p(\mu_{p^n}) / \mathbb{Q}_p$

Ⓑ $(p^f) \times \mathcal{U}_{\mathbb{Q}_p} \longleftrightarrow$ unramified ext'n of deg. f.

$\Longrightarrow M \subseteq \mathbb{Q}_p\left(\zeta_{(p^f-1)p^n} \right)$

(*) Fact that every open subgp N of finite index in $K^\times$ contains such a subgp. of form $(\pi^f) \times \underbrace{\mathcal{U}_K^{(n)}}_{\text{open sbgp. of identity}}$, $n \geq 0$, $f \geq 0$.

$1 + \mathfrak{p}_K^n$ and set $\mathcal{U}_K^{(0)} = \mathcal{U}_K$.

<u>Prop</u>: L/K finite, abelian ext'n is unramified $\iff$ $\mathcal{U}_K \subseteq N_{L/K} L^\times$

$(\Leftarrow)$ if M/K unramified, degree n, then $N_{M/K} M^\times = (\pi_K^n) \times \mathcal{U}_K \subseteq N_{L/K} L^\times$ by assumption.

so $L \subseteq M$, i.e. unramified.

$(\Rightarrow)$ L/K unram. $\Rightarrow \mathcal{U}_K = N_{L/K} \mathcal{U}_L$. //

Thus in case Ⓑ above, know corresponding field is unramified.

But result Ⓐ is specific to $\mathbb{Q}_p$.

proof of Corollary 2 from Corollary 1 : $S :=$ finite set of primes ramified in L.

Let L_p : completion of L w.r.t. prime above $p \in S$. As

$$\begin{tikzcd}[row sep=small, column sep=small] & L_p = L \cdot \mathbb{Q}_p \arrow[d, dash] \\ L \arrow[ur, dash] \arrow[d, dash] & \mathbb{Q}_p \\ \mathbb{Q} \arrow[ur, dash] & \end{tikzcd}$$

then $L_p / \mathbb{Q}_p$ is abelian ext'n, so by

Corollary 1 $L_p \subseteq \mathbb{Q}_p(\mu_{n_p})$ for some $n_p \in \mathbb{N}$. Let $p^{e_p} \| n_p$

and set $n = \prod_{p \in S} p^{e_p}$. Claim: $L \subseteq \mathbb{Q}(\mu_n)$.

pf of claim: Set $M = L(\mu_n)$ Then $M/\mathbb{Q}$ abelian with p ramified in $M/\mathbb{Q}$

only if $p \in S$. M_p completion of M restricting to L_p.

So $M_p = L_p(\mu_n) = \mathbb{Q}_p(\mu_{p^{e_p}}) \cdot \mathbb{Q}_p(\mu_{n'})$ where $n = p^{e_p} n'$ with $\gcd(n', p) = 1$

Inertia subgp. I_p of $M_p/\mathbb{Q}_p$ is $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^{e_p}})/\mathbb{Q}_p)$

since $\mathbb{Q}_p(\mu_{n'})$ is maximal unram. ext'n of $\mathbb{Q}_p(\mu_n)$

so I_p has order $\varphi(p^{e_p})$. Let $I = \langle I_p \rangle_{p \in S} \subseteq \mathrm{Gal}(M/\mathbb{Q})$

so that fixed field of I is unramified ext'n of $\mathbb{Q}$.

By a thm. of Minkowski, only unramified ext'n of $\mathbb{Q}$ is $\mathbb{Q}$ itself.

So $I = \mathrm{Gal}(M/\mathbb{Q})$. Thus $\# I = \# \mathrm{Gal}(M/\mathbb{Q})$
$\geq [\mathbb{Q}(\mu_n) : \mathbb{Q}]$.

On other hand:

$$\# I \leq \prod_{p \in S} \# I_p = \prod_{p \in S} \varphi(p^{e_p}) = \varphi(n) = [\mathbb{Q}(\mu_n) : \mathbb{Q}] \quad //$$

In general, we'd like to know class fields for important set of open subgps $(\pi^f) \times \mathcal{U}_K^{(n)}$. Theory of Lubin–Tate extensions.

These are extensions $L_n = K(F(n))$ where $F(n)$ evaluation of formal gp. law F corresponding to π.
(for $f=1$)

elts killed by π^n in alg. closure.
mult. is formal gp. mult.

Take composition with unram. ext'n corresponding to $(\pi^f) \times \mathcal{U}_K$ to get open subgp. $(\pi^f) \times \mathcal{U}_K^{(n)}$.

Examples: Ask for primes p s.t. f splits completely into linear factors where $K_f/\mathbb{Q}$ having minimal poly f

Theorem: $K_f \supset K_g$ iff $Spl(f)$: primes split in K_f $\subseteq^* Spl(g)$

where $\subseteq^*$ means "up to finitely many exceptions"

$(\Rightarrow)$ easy. Multiplicativity of e's, f's in towers.

$(\Leftarrow)$ Tchebotarev density thm

explicit realization of $\sqrt{p}$ in $\mathbb{Q}(\zeta_p)$ by Gauss sums.

Example: $K_g = \underline{\mathbb{Q}(\sqrt{p})}$ $\quad K_f = \underline{\mathbb{Q}(\zeta_p)}$ $\quad p \equiv 1\ (4)$

q splits completely $\Leftrightarrow \left(\frac{q}{p}\right) = 1$

q splits completely $\Leftrightarrow q \equiv 1\ (p)$
[then residue field contains p^{th} roots of unity

so indeed $K_f \supseteq K_g$ since $Spl(f) \supset Spl(g)$