Name: ______________________________

Problem Set 12
Math 4281, Fall 2013
Due: Monday, December 2

Read Sections 6.1, 6.2, 10.1, and 11.2 through Example 8, as well as Theorem 11.2 in your textbook.

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1. Show that H is a normal subgroup of G, but K is not a normal subgroup of G, where
$$\begin{aligned} G &= \left\{ \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ -1 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & -1 \\ 0 & 1 \end{smallmatrix}\right] \right\}, \\ H &= \left\{ \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right] \right\}, \quad \text{and} \\ K &= \left\{ \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \right\}. \end{aligned}$$

2. Prove that $Z = \{a \in G \mid ax = xa \text{ for all } x \in G\}$ is a normal subgroup of G. (This is called the **center** of G.)

3. Let G be a group with identity element e. Suppose that $H, K \subseteq G$ are subgroups of orders 5 and 8, respectively. Prove that $H \cap K = \{e\}$.

4.
   a. Prove that a group G of even order has an element of order 2. (Hint: If $a \neq e$, then a has order 2 if and only if $a = a^{-1}$.)
   b. Suppose m is odd, $|G| = 2m$, and G is abelian. Prove G has precisely one element of order 2. (Hint: If there were two, they would provide a Klein four-group.)
   c. Prove that if G has exactly one element of order 2, then it must be in the center of G.

5. Let G be a group and $H \leq G$.
   a. Prove that H is a normal subgroup of G if and only if every left coset is a right coset, i.e., $aH = Ha$ for all $a \in G$.
   b. Use (a) to show that if $[G : H] = 2$, then H is a normal subgroup of G.

6. Let $H \subseteq K \subseteq G$ be subgroups. Prove that
$$[G : H] = [G : K][K : H].$$
(When G is infinite, the statement includes the observation that $[G : H]$ is finite if and only if $[G : K]$ and $[K : H]$ are both finite.)

Through the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.

Signed: ______________________________