Math 5385 - Spring 2018
Problem Set 10

Submit solutions to **three** of the following problems.

1. Let $I, J \subseteq \Bbbk[x_1, \ldots, x_n]$ be ideals.
   (a) Prove that $I : J^\infty = I : J^N$ if and only if $I : J^N = I : J^{N+1}$. Then use this to describe an algorithm for computing the saturation $I : J^\infty$ based on the algorithm for computing ideal quotients.
   (b) Show that N can be arbitrarily large in $I : J^\infty = I : J^N$. **Hint.** Try $I = \langle x^N(y-1)\rangle$.
2. (a) Consider $X = V(x^2 - yz, xz - x) \subseteq \mathbb{A}^3(\Bbbk)$. Show that X is a union of three irreducible components. Describe them and find their prime ideals.
   (b) Show that the set of real points on the irreducible complex surface
   $$V\left((x^2 + y^2)z - x^3\right) \subset \mathbb{A}^3(\mathbb{C}).$$
   is connected but is not equidimensional; it is the union of a closed curve and a closed surface in the induced Euclidean topology.
3. (a) Show that the intersection of any collection of prime ideals is radical.
   (b) Show that an irredundant intersection of at least two prime ideals is never prime.
4. Let $I = \langle xz - y^2, x^3 - yz\rangle$.
   (a) Show that $I : (x^2y - z^2) = \langle x, y\rangle$.
   (b) Show that $I : (x^2y - z^2)$ is prime.
   (c) Show that $I = \langle x, y\rangle \cap \langle xz - y^2, x^3 - yz, z^2 - x^2y\rangle$.
5. Let $I = \langle xz - y^2, z^3 - x^5\rangle \subseteq \mathbb{Q}[x, y, z]$.
   (a) Express $V(I)$ as a finite union of irreducible varieties.
   Hint. The parametrizations (t^3, t^4, t^5) and $(t^3, -t^4, t^5)$ will be useful.
   (b) Express I as an intersection of prime ideals which are ideal quotients of I and conclude that I is radical.