Math 5385 - Spring 2018
Problem Set 3

Submit solutions to **three** of the following problems.

1. Here we study the *consistency problem* from §1.2 in the one-variable case. Given $f_1, \ldots, f_s \in \Bbbk[x]$, this asks if there is an algorithm to decide if $V(f_1, \ldots, f_s)$ is nonempty. You will show that the answer is yes when $\Bbbk = \mathbb{C}$.
   (a) Let $f \in \mathbb{C}[x]$ be a nonzero polynomial. Use Theorem 1.1.7 to show that $V(f) = \varnothing$ if and only if f is constant.
   (b) If $f_1, \ldots, f_s \in \mathbb{C}[x]$, prove $V(f_1, \ldots, f_s) = \varnothing$ if and only if $\gcd(f_1, \ldots, f_s) = 1$.
   (c) Describe (in words, not pseudocode) an algorithm for determining whether or not $V(f_1, \ldots, f_s) \subseteq \mathbb{A}^n(\mathbb{C})$ is nonempty.

   When $\Bbbk = \mathbb{R}$, the consistency problem is much more difficult. It requires giving an algorithm that tells whether a polynomial $f \in \mathbb{R}[x]$ has a real root.

2. Suppose that $\Bbbk$ is an infinite field. Let $X \subset \mathbb{A}^3(\Bbbk)$ be the set $X = \{(t, t^2, t^3) \mid t \in \mathbb{A}^1(\Bbbk)\}$.
   (a) Use the parametrization of X to show that $z^2 - x^4 y$ vanishes at every point.
   (b) Find a representation $z^2 - x^4 y = h_1(y - x^2) + h_2(z - x^3)$, where $h_1, h_2 \in \Bbbk[x, y, z]$.
   (c) Use the division algorithm to show that $I(X) = \langle y - x^2, z - x^3 \rangle$.

3. Let $I = \langle x^{u_1}, \ldots, x^{u_p} \rangle$ and $J = \langle x^{v_1}, \ldots, x^{v_q} \rangle$ be two monomial ideals in $S = \Bbbk[x_1, \ldots, x_n]$.
   (a) If x^w is a monomial in S, then prove that the ideal $(I : x^w) := \{f \in S \mid f x^w \in I\}$ is generated by the monomials of $x^{u_i} / \gcd(x^{u_i}, x^w)$ for $1 \leq i \leq p$.
   (b) Show that $I \cap J$ is generated by monomials $\mathrm{lcm}(x^{u_i}, x^{v_j})$ for $1 \leq i \leq p$ and $1 \leq j \leq q$.

4. Assume that $x_1 > x_2 > \cdots > x_n$. Show that the following properties characterize the monomial orders $>_{\mathrm{lex}}$ and $>_{\mathrm{grevlex}}$ among all monomial orders on $S = \Bbbk[x_1, \cdots, x_n]$.
   (a) If $\mathrm{LT}_{\mathrm{lex}}(f) \in \Bbbk[x_i, \ldots, x_n]$ for some $1 \leq i \leq n$, then $f \in \Bbbk[x_i, \ldots, x_n]$.
   (b) The monomial order $>_{\mathrm{grevlex}}$ refines the partial order given by total degree and, for homogeneous f, the condition $\mathrm{LT}_{\mathrm{grevlex}}(f) \in \langle x_i, \ldots, x_n \rangle$ for some $1 \leq i \leq n$ implies that $f \in \langle x_i, \ldots, x_n \rangle$.

5. Let M be an $(m \times n)$-matrix with nonnegative real entries and let $r_1, \ldots, r_m$ denote the rows of M. Assume that $\ker(M) \cap \mathbb{Z}^n = \{0\}$. Define a binary relation $>_M$ on the monomials in $S = \Bbbk[x_1, \ldots, x_n]$ as follows: $x^u >_M x^v$ if there is a $\ell \leq m$ such that $u \cdot r_i = v \cdot r_i$ for all $1 \leq i \leq \ell - 1$ and $u \cdot r_\ell > v \cdot r_\ell$.
   (a) Show that $>_M$ is a monomial order on $S = \Bbbk[x_1, \ldots, x_n]$.
   (b) If $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then $>_M$ equals $>_{\mathrm{grevlex}}$ on $\Bbbk[x, y, z]$.
   (c) If I is the $(n \times n)$-identity matrix, then show that $>_{\mathrm{lex}}$ equals $>_I$.