Math 8212 Homework

You may collaborate on the homework. For the entire semester, you will work in a small group of 1 to 4 people. Email your list of group members to me by 5 p.m. on January 22. From this list of problems, your group should submit 20 problems typed in LaTeX, which will be collected together on the last day of class, May 2, in a hard copy; an electronic copy should also be sent to my email address at that time. Clearly number the problem with the numbers below, as well as their placement in Eisenbud (if applicable). In placing your name on an assignment with the others in your group, you are agreeing that each person listed has made a substantial contribution to and agrees with the solutions provided.

(1) Eisenbud, Exercise 9.1.
(2) Eisenbud, Exercise 9.2.
(3) Eisenbud, Exercise 9.3.
(4) Eisenbud, Exercise 9.5 or Exercise 9.6.
(5) Verify explicitly that Krull dimension satisfies the first half of Axiom D1, i.e.,

$$\dim R = \sup_{P \subseteq R \text{ prime}} \dim R_P$$

and Axiom D2, that $\dim R = \dim R/I$ if I is a nilpotent ideal.
(6) Eisenbud, Exercise 10.2.
(7) Eisenbud, Exercise 10.3.
(8) Eisenbud, Exercise 10.8.
(9) Find an example of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{m}$ has four generators, but $\dim R = 2$. Find an explicit system of parameters $\{f_1, f_2\}$ for R and prove that this is a system of parameters. Is it also a regular sequence?
(10) Consider the ring $k[x, y, z]/(xz, yz)$.
- Show that the ring is 2-dimensional, and find an explicit system of parameters.
- Prove that the ring does not have any regular sequence $f_1, f_2 \subseteq \mathfrak{m}$.

(11) Challenge problem: Eisenbud, Exercise 10.9.
(12) Eisenbud, Exercise 10.4.
(13) Eisenbud, Exercise 10.6.
(14) Eisenbud, Exercise 11.1.
(15) Eisenbud, Exercise 11.3.
(16) Eisenbud, Exercise 11.7.
(17) Eisenbud, Exercise 13.2 (this one comes with a hint).
(18) Eisenbud, Exercise 13.3 (this one also comes with a big hint).
(19) Eisenbud, Exercise 13.9.
(20) Theorem 13.3 (well it's basically Lemma 13.2(c)) guarantees that over an infinite field, you can always find a "linear" Noether normalization. Produce a counterexample to this over a finite field as follows. Let $k = \mathbb{Z}/2$ and let $S = k[x, y]$. Find a polynomial $f \in k[x, y]$ such that the ring $R := S/(f)$ does not admit a linear Noether normalization.
(21) Let $I = \langle xz - y^2, yw - z^2, yz - xw \rangle \subseteq \mathbb{Q}[x, y, z, w]$ and let $f = x^2y^2w^2 - y^4z^2$. Use the division algorithm (by hand!) to determine whether or not f lies in I.
(22) We use the notation $f \% g$ to denote the remainder of f when divided by g using the division algorithm. (This is the syntax for Macaulay2 as well.) Over $\mathbb{Q}[x, y, z]$ find

an example of polynomials f, g_1, and g_2 where

$$(f \ \% \ g_1) \ \% \ g_2 \neq (f \ \% \ g_2) \ \% \ g_1.$$

Bonus: Can you find f, g_1, g_2 such that $(f \ \% \ g_1) \ \% \ g_2 = 0$ but $(f \ \% \ g_2) \ \% \ g_1 \neq 0$?

(23) Find an example of an ideal I where I is generated by quadrics by where $\mathrm{in}_>(I)$ has a minimal generator of degree at least 4. You can choose any monomial order $>$ you like.

(24) Let $>$ be a monomial ordering on S that respects degree, i.e., $x^\alpha > x^\beta$ whenever $|\alpha| > |\beta|$. For an arbitrary (i.e., not necessarily homogeneous) polynomial $g \in S := k[x_1, \ldots, x_n]$ of degree d, we define the homogenization of g by x_0 as

$$g^h(x_0, x_1, \ldots, x_n) := x_0^d \cdot g(\tfrac{x_1}{x_0}, \ldots, \tfrac{x_n}{x_0}).$$

For an ideal $I \subseteq S$ define

$$I^h := \langle g^h | g \in I \rangle \subseteq k[x_0, \ldots, x_n].$$

Fix a monomial ordering $>$ on S, and let $\mathcal{G} = \{g_1, \ldots, g_r\}$ be a Gröbner basis of I. Prove that

$$I^h = \langle g_1^h, \ldots, g_r^h \rangle.$$

(25) Using the lex term order, compute a reduced Gröbner basis (by hand!) for the ideal

$$\langle xy - x - 2y + 2, x^2 + xy - 2x \rangle \subseteq \mathbb{C}[x, y].$$

Use this to determine all solutions in $\mathbb{C}^2$ to the system of equations:

$$\begin{cases} xy - x - 2y + 2 & = 0, \\ x^2 + xy - 2x & = 0. \end{cases}$$

Bonus: instead of $\mathbb{C}$, we could have worked over a field k of positive characteristic. Over which characteristics would the Gröbner basis have been the same?

(26) Given two complexes $F_\bullet$ and $G_\bullet$, we can define a new complex by

$$(F \otimes G)_i := \bigoplus_{p+q=i} F_p \otimes G_q$$

with boundary

$$\partial(f \otimes g) = \partial f \otimes g + (-1)^p f \otimes \partial g.$$

Fix a (commutative, Noetherian) ring R and $x_1, x_2, x_3 \in R$. Confirm that, with this definition, there exists a natural isomorphism (up to signs):

$$\mathbf{K}(x_1, x_2) \otimes \mathbf{K}(x_3) \cong \mathbf{K}(x_1, x_2, x_3).$$

(27) Eisenbud, Exercise 17.1.
(28) Eisenbud, Exercise 17.2.
(29) Eisenbud, Exercise 17.3.
(30) Eisenbud, Exercise 19.7.
(31) Eisenbud, Exercise 18.2.
(32) Eisenbud, Exercise 18.6.
(33) Eisenbud, Exercise 19.13.
(34) Eisenbud, Exercise 20.6.
(35) Eisenbud, Exercise 20.16.